More information about Homology

By: Smooth Endoplastic Reticulum

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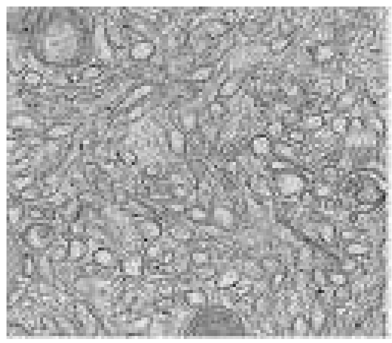

Hello med students! This is a picture of smooth endoplastic reticulum here to finish off the proof that Corey asked you about in his homework assignment to you all. See, I sort of thought that there was enough info regarding abstract algebra to warrant a handout, so here you are! Corey is hard at work on your next assignment and expects to have it done soon. But he's been working very hard lately and sort of deserves a break. So he went outside yesterday and got an ENORMOUS sunburn on his legs. Yes. But, as a piece of endoplastic reticulum, as we say, "everyone knows there aren't enough O's in SMOOOOOOTH!" ROCK ON!

Let's start out with a theorem that is not too hard to prove, but we include a proof just to be nice:

Theorem 0.1 *Suppose G_1 and G_2 are abelian groups, and $H_1 \leq G_1, H_2 \leq G_2$. Then*

$$\frac{G_1}{H_1} \oplus \frac{G_2}{H_2} \cong \frac{G_1 \oplus G_2}{H_1 \oplus H_2}.$$

That is, fractions of groups don't work the same way as fractions of numbers! Before we start, it is worth noting that this all makes sense in the first place. Since everything is abelian, all subgroups are normal, and thus the quotient sets G_i/H_i are actually groups. This also applies to the subgroup $H_1 \oplus H_2 \leq G_1 \oplus G_2$. I just sort of think it's worthwhile to point out the subtle differences in the group structures above, even though in a formal proof I may not mention this stuff. Also, when you see the definition of φ below, I really think that this is an accurate choice of notation to use–in particular, I'm talking about expressing elements of quotient groups in terms of their coset representation. The formula for φ is nothing complicated at all, it's just wading through the notation, and I think by pointing this stuff out before I get going it may help you understand the short proof as a whole.

Proof. Let $\varphi : \frac{G_1}{H_1} \oplus \frac{G_2}{H_2} \to \frac{G_1 \oplus G_2}{H_1 \oplus H_2}$ be defined as $\varphi(g_1 + H_1, g_2 + H_2) = (g_1, g_2) + H_1 \oplus H_2$. We will show that φ is (1) well-defined (which is necessary since we are defining its rule on representatives of equivalence classes), (2) a homomorphism, (3) one-to-one, and (4) onto.

(1) Suppose that $g_1 + H_1 = \tilde{g}_1 + H_1$, for $i = 1, 2$. Then there exists an $h_1 \in H_1$ so that $\tilde{g}_1 = g_1 + h_1$. So

$$\begin{aligned}
\varphi(\tilde{g}_1 + H_1, g_2 + H_2) &= (\tilde{g}_1, g_2) + H_1 \oplus H_2 \\
&= (g_1 + h_1, g_2) + H_1 \oplus H_2 \\
&= (g_1, g_2) + (h_1, 0) + H_1 \oplus H_2, \text{ and } (h_1, 0) \in H_1 \oplus H_2, \text{ so} \\
\varphi(\tilde{g}_1 + H_1, g_2 + H_2) &= (g_1, g_2) + (h_1, 0) + H_1 \oplus H_2 \\
&= (g_1, g_2) + H_1 \oplus H_2 \\
&= \varphi(g_1, g_2)\,.
\end{aligned}$$

The exact same argument is used to show that φ is well-defined on the other summand.

(2) Let $g_i, \tilde{g}_i \in G_i$. Then

$$\begin{aligned}
\varphi(g_1 + \tilde{g}_1 + H_1, g_2 + \tilde{g}_2 + H_2) &= (g_1 + \tilde{g}_1, g_2 + \tilde{g}_2) + H_1 \oplus H_2 \\
&= (g_1, g_2) + (\tilde{g}_1, \tilde{g}_2) + H_1 \oplus H_2 \\
&= \varphi(g_1 + H_1, g_2 + H_2) + \varphi(\tilde{g}_1 + H_1, \tilde{g}_2 + H_2)\,.
\end{aligned}$$

(3) Suppose $\varphi(g_1 + H_1, g_2 + H_2) = (0, 0) + H_1 \oplus H_2$, that is, suppose $(g_1 + H_1, g_2 + H_2) \in \ker \varphi$. Then $\varphi(g_1 + H_1, g_2 + H_2) \in H_1 \oplus H_2$, so $(g_1, g_2) \in H_1 \oplus H_2$, and so $g_i \in H_i$, and the kernal of φ is trivial.

(4) This map is clearly onto: the arbitrary element $(g_1, g_2) + H_1 \oplus H_2$ is the image of the element $(g_1 + H_1, g_2 + H_2)$.

So this map φ is an isomorphism.

□

We will use this information to pound the following fact into the ground: That if X is the disjoint union of the sets A and B, then

$$H_n(A \sqcup B) \cong H_n(A) \oplus H_n(B)\,.$$

We will do this in several steps, which are outlined in this Lemma:

Lemma 0.2 *Use the notation we've developed in the class so far. We omit the subscript on the ∂_n operator unless there is any cause for confusion.*

1. *The homomorphism* $\partial : C_n(A) \to C_{n-1}(A)$ *and* $\partial : C_n(B) \to C_{n-1}(B)$.

2. $C_n(A \sqcup B) \cong C_n(A) \oplus C_n(B)$.

3. $\ker \partial \cong \ker \partial|_{C_n(A)} \oplus \ker \partial|_{C_n(B)}$.

4. $\mathrm{Im}\ \partial \cong \mathrm{Im}\ \partial|_{C_{n+1}(A)} \oplus \mathrm{Im}\ \partial|_{C_{n+1}(B)}$.

5. $H_n(A \sqcup B) \cong H_n(A) \oplus H_n(B)$.

Proof. I won't type out the proof of (1), since this was done in class. Morally speaking, though, the reason why this is true is because the standard n-simplex is connected, and any $\sigma : \Delta_n \to X$ must have a connected image.

We also sort of did (2) in class. We decided that $C_n(A) + C_n(B) = C_n(A \sqcup B)$, that is, that $C_n(A \sqcup B)$ was generated by elements in both $C_n(A)$ and $C_n(B)$. We also decided that there were no chains that were both in $C_n(A)$ and $C_n(B)$ since such a chain would have a singular simplex with a disconnected image.

The proofs of (3) and (4) are straightforward, as both $\ker \partial$ and $\mathrm{Im}\ \partial$ are subgroups of the group $C_n(A) \oplus C_n(B)$. We use the previous theorem to finish the proof, since is establishes that

$$H_n(A \sqcup B) = \frac{\ker \partial}{\mathrm{Im}\ \partial} \cong \frac{\ker \partial|_{C_n(A)} \oplus \ker \partial|_{C_n(B)}}{\mathrm{Im}\ \partial|_{C_{n+1}(A)} \oplus \mathrm{Im}\ \partial|_{C_{n+1}(B)}}$$

$$\overset{\text{(by thm)}}{\cong} \quad \frac{\ker \partial|_{C_n(A)}}{\mathrm{Im}\ \partial|_{C_{n+1}(A)}} \oplus \frac{\ker \partial|_{C_n(A)}}{\mathrm{Im}\ \partial|_{C_{n+1}(A)}} = H_n(A) \oplus H_n(B)\,.$$