

Math 555 midterm solutions!!!

Shake

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Hi kids! Here are your solutions. Now, I'm going to go deal with those pretentious Mooninites, and then go over to Carl's pool! ROCK ON!

1. Corey suggests you comb through the book and the notes for these definitions.
2. (a) Notice that by hypothesis, the set $V = \cup_{x \in A} U_x \subseteq A$, since each $U_x \subseteq A$. Notice also that $A \subseteq V$, since every $x \in A$ is assumed to also be in the sets U_x . Thus $V = A$. Since each U_x is open, we conclude that V , as the union of open sets, is open.
(b) Corey went over this fact carefully in class, see the class notes for a solution to this one.
(c) Corey also covered this problem in detail in class. See your class notes for a solution.
3. (a) Corey covered this problem in class as a theorem, and proved the result there. Consult your notes for a solution.
(b) Suppose there exists $U \subseteq X$, so that U is open and closed. Define $V = U^c$. Since U is closed, V is open. Now the sets U and V are both open, and satisfy $U \cup V = X$, and $U \cap V = \emptyset$.
(c) This is a previous homework problem, and the solution exists as part of our discussion on it in class.

- (d) We apply Theorem 13.2. First, each of the sets in \mathcal{B} is open (in the standard topology). Second, for all $x \in \mathbb{R}$, there exists rational numbers $a < x < b$, so each real number has a set in \mathcal{B} in which it fits. Further, for any open set U of \mathbb{R} , and $x \in U$, there exists a set (α, β) , where $\alpha, \beta \in \mathbb{R}$, and $x \in (\alpha, \beta)$. Since \mathbb{Q} is dense in \mathbb{R} , there exist $a, b \in \mathbb{Q}$ with $x \in (a, b) \subseteq (\alpha, \beta)$. So by Theorem 13.2, this collection is a basis for the standard topology on \mathbb{R} .
4. (a) (i) The particular point topology is a non-Hausdorff topology on the real number line.
(ii) See the solutions to homework number 3 for the solution to this problem.
(b) See the proof of 17.8 for a solution. Also, this was proved in class.
(c) This fact was proved in class, and is a homework problem.
(d) (i) Any closed subset of the real numbers works.
(ii) Use $A = \mathbb{Z}$.
(iii) Consider $A = \mathbb{R} - \{0\}$.
(iv) Use $A = \mathbb{Q}$.
(v) The answer is no. Enumerate a countable sequence of elements from such a set A . The Bolzano-Weierstrass Theorem says that this bounded sequence has a convergent subsequence. The limit of this convergent subsequence is a limit point, thus every infinite bounded subset of the real numbers has a limit point. (See also, for an exact statement of this precise question, the Bolzano-Weierstrass Theorem for sets, found in almost any analysis text.)
5. (a) and (b) were proved in class. Part (c) was proved in the handout of things Corey gave you that he wanted to type out.
(d) (i) Suppose $x_n \rightarrow x$ in the $\epsilon - N$ definition. Let U be a neighborhood of x . Thus, there exists an ϵ for which $x \in B_\epsilon(x) \subseteq U$. Therefore, there exists an N for which $x_n \in B_\epsilon(x)$ for all $n \geq N$. And so for a neighborhood U of x , we found an N for which $x_n \in U$ for all $n \geq N$.
Now, conversely, assume that for all neighborhoods U of x , there exists an N for which $x_n \in U$ for all $n \geq N$. Let $\epsilon > 0$ be given, and choose our neighborhood to be $U = B_\epsilon(x)$. Then we can find N for which $x_n \in B_\epsilon(x)$ for all $n \geq N$.
(ii) Consider any space with more than two elements, endowed with the trivial topology. It follows that *every* sequence converges to *every* element, since the only neighborhood of any element is the whole space, which contains the entire sequence.