

# HW # 4 Solutions

Batman, Superman, and Robin

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*Hi everyone, this is Superman, who needed my buddy Batman and his sidekick Robin to help with this week's homework solutions. For some reason my super body and lightning quick reflexes can't help me solve topology homework questions—could this be another kryptonite? Corey recalls doing number 3 from the sheet in class, and also recalls asking you the false question number 4, which he has revised and put on the next homework assignment. The rest are from the book. ROCK ON!*

2. The answer is no. Consider the constant function  $f(x) = 5$ , where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Then every point  $x \in \mathbb{R}^2$  is a limit point, although the one point set  $f(\mathbb{R}^2) = \{5\}$  contains no limit points. So use  $A = \mathbb{R}^2$ .
5. We observe that the composition of homeomorphisms is a homeomorphism. This follows for the following reasons: if  $f : X \rightarrow Y$  is a homeomorphism, and  $g : Y \rightarrow Z$  is also a homeomorphism, then:
  - (a) Both  $f^{-1}$  and  $g^{-1}$  are homeomorphisms, since  $(f^{-1})^{-1} = f$  and  $(g^{-1})^{-1} = g$ .
  - (b)  $g \circ f$  is a bijection since the composition of bijections is a bijection.
  - (c)  $g \circ f$  is continuous, since the composition of continuous functions is continuous.
  - (d)  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ , so  $(g \circ f)^{-1}$  is both bijective and continuous as compositions of bijective and continuous functions.

So we construct a homeomorphism  $f : [0, 1] \rightarrow [0, b - a]$ , then a homeomorphism  $g : [0, b - a] \rightarrow [a, b]$ , then the composition will be a homeomorphism from  $[0, 1]$  to  $[a, b]$ .

So set  $f(x) = (b - a)x$ . Since  $b > a$ , this is obviously continuous and has an inverse which is continuous (we call such a function “bi-continuous”). Set  $g(x) = x + a$ , so that  $g$  is obviously bi-continuous as well.

Notice that  $(g \circ f)(0) = a$  and  $(g \circ f)(1) = b$ , so the restriction  $(g \circ f)|_{(0,1)}$  is bijective and bicontinuous onto  $(a, b)$ , completing the problem.

6. Corey sort of hinted that

$$f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q} \end{cases}$$

is only continuous at  $x = 0$ . To prove this, we use the  $\epsilon - \delta$  notion of continuity, which is equivalent to our topological notion of continuity in  $\mathbb{R}$ . If  $x = 0$ , then we see that if  $\delta = \epsilon$ , that this establishes continuity there. If  $x \neq 0$ , then we may choose  $\epsilon = |x|/2$ , and it follows that every neighborhood of  $x$  will contain values  $f(x)$  which are either 0 or  $x$  and fall outside the given  $\epsilon$ -neighborhood of  $f(x)$ . So  $f$  is not continuous at any other point.

13. Corey seems to remember doing this in class, and that Stewart bailed him out when he started to space out.