Math 555 Homework # 2

Mr. Burns

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I've spent some time working out solutions to your homework, and reluctantly make them available to you! Okay. Not reluctantly at all. Ehhhhhhxxxxxcccellent. ROCK ON.

1. Starting from page 91, please do problems 1, 3, 4 and 6.

   1. Suppose $A \subseteq Y \subseteq X$. Let $\tau_{A,Y}$ be the topology A inherits as a subspace of Y, and let $\tau_{A,X}$ be the topology A inherits as a subspace of Y. We are to show that $\tau_{A,Y} = \tau_{A,X}$, and we do this by double-inclusion.

   First let $U \in \tau_{A,Y}$. Then there is an open set U_Y in Y (endowed with the subspace topology…this is an assumption that is implicit, not a fact that follows from what I just typed), so that $U = U_Y \cap A$. But U_Y is open in Y, so there is an open set U_X in X so that $U_Y = U_X \cap Y$. Then $U = (U_Y) \cap A =$

$(U_X \cap Y) \cap A = U_X \cap (Y \cap A)$. But $A \subseteq Y$, so $A \cap Y = A$, and so $U = U_X \cap A$, the intersection of an open set of X intersected with A. So $U \in \tau_{A,X}$.

Now let $U \in \tau_{A,X}$. Then there is an open set U_X of X so that $U = U_X \cap A$. But again, since $A \subseteq Y$, $U_X \cap A = U_X \cap Y \cap A$, and we have exhibited $U = (U_X \cap Y) \cap A$ as the intersection of the open set $U_Y = U_X \cap Y$ of Y intersected with A, so $U \in \tau_{A,Y}$.

3 and 4. Corey suggests that he did these in class, and excuses himself from typing out detailed solutions. I tell him loudly, "YOU'RE FIRED!!!"

6. There is a minor amount of ambiguity here, although Corey mentioned in class how to deal with this. The question asks you to show that a certain collection is a basis for a topology. That's just an application of the definition of basis. What Corey thinks the problem actually *could* ask additionally is if the topology generated by this basis coincides with the standard topology on $\mathbb{R}^2$. We'll answer both here.

First we check that the given collection is a basis. The first requirement is easy to check: let $(x, y) \in \mathbb{R}^2$. There exists rational numbers a, b, c, and d so that $a < x < b$ and $c < y < d$, so that the basis element $(a, b) \times (c, d)$ contains (x, y). Now suppose that there is an element in the intersection

$$(a, b) \times (c, d) \cap (p, q) \times (r, s) = [(a, b) \cap (p, q)] \times [(c, d) \cap (r, s)].$$

That there exists an element in the intersection implies that each of the intersected factors $[(a, b) \cap (p, q)]$ and $[(c, d) \cap (r, s)]$ are nonempty. Thus,

$$\begin{aligned} (a, b) \cap (p, q) &= (p, b) \text{ or } (a, q), \text{ and} \\ (c, d) \cap (r, s) &= (r, d) \text{ or } (c, s). \end{aligned}$$

In either case, these basis elements intersect to *another* basis element. It now follows that this collection is a basis.

Now, we show this basis coincided with the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (we already showed in class that the product topology on $\mathbb{R}^n$ coincides with the standard topology on $\mathbb{R}^n$). Note that given $(x, y) \in \mathbb{R}^2$, and *real* numbers $\alpha, \beta, \delta, \gamma$, with $(x, y) \in (\alpha, \beta) \times (\delta, \gamma)$, there exist *rational* numbers a, b, c, and d with $\alpha < a < x < b < \beta$ and $\delta < c < y < d < \gamma$ (this follows by four applications of the denseness of $\mathbb{Q}$ in $\mathbb{R}$). Thus

$$(x, y) \in (a, b) \times (c, d) \subseteq (\alpha, \beta) \times (\delta, \gamma).$$

The other basis containment is completely obvious, since every rational number is also a real number. So these bases generate the same topology on $\mathbb{R}^2$.

Corey hastens to point out that for every single question he asks to you, there is a specific purpose. The purpose of asking you *this* question is that one can do with a *much* smaller basis than the one given by the product topology on $\mathbb{R}^2$ (which follows from the easier statement about the basis for the standard topology on $\mathbb{R}$, see Theorem 15.1 for the connection). In fact, this basis is *countable* in size, instead of the other standard basis which is clearly uncountable. Corey thinks that property is actually something that topologists care about and thinks one would call a space with a countable basis *second countable*. If you're wondering what would make a space *first countable*, Corey urges you to consult page 190 in our book.

2. (a) Suppose that X is a topological space, and that $A \subseteq Y \subseteq X$. Prove that if A is open in the subspace topology of Y, and Y is open in X, then A is open in X. The justifies the statement "open subsets of open subsets are open." Isn't that cute?

   (b) Suppose again that $A \subseteq Y \subseteq X$, and that A is open in Y. But do *not* assume that Y is open in X. Give an example of a situation where A is open, an example where A is closed, and an example of when A is neither open or closed. (This may require you to possibly choose different Y as well.)

   Corey also seems to recall going over this in class, but would be happy to discuss any of these further. But for now, we can let him off the hook for typing them out.