

HW # 1

Brian Griffin

October 8, 2008



Hi kids. I worked on your assignment, and here are some solutions. I won't restate the hypotheses unless it adds emphasis to my presentation, although I do not necessarily suggest that you omit the statement of the problem in your own work (or restate hypotheses in your solutions). ROCK ON!

1. Problem 1 on page 83: Let $x \in A$. By hypothesis, there is an open set $U_x \subseteq A$ with $x \in U_x$. Consider the set $\cup_{x \in A} U_x$. Since each $U_x \subseteq A$, we have $\cup_{x \in A} U_x \subseteq A$. But for each $x \in A$, there is a set U_x with $x \in U_x$. Thus $A \subseteq \cup_{x \in A} U_x$. By double inclusion, we see that $A = \cup_{x \in A} U_x$, and so, as the arbitrary union of open sets, A is open.

Problem 8 on page 84: We'll do part (a), with the understanding that part (b) won't be graded. To show that \mathcal{B} is a basis which generates the standard topology on \mathbb{R} , we have to show that \mathcal{B} is a basis, and that it generates the same topology on \mathbb{R} as the standard topology.

\mathcal{B} is a basis: First, let $x \in \mathbb{R}$. Clearly, there exist rational numbers $a, b \in \mathbb{Q}$ so that $a < x < b$, so that the interval $x \in (a, b) \in \mathcal{B}$. Second, when one intersects two intervals, one of three things happen. First, the intersection could be empty. Second, one interval is a subset of the other. Lastly, neither interval is a subset of each other. The first case is not interesting, as there are no elements in that intersection. The second case is not interesting either, since the intersection of two such intervals would be one of the intervals. Thus, the condition to be a basis is easily satisfied. If (a, b) and (c, d) are elements of \mathcal{B} which are not subsets of each other, then the rational numbers a, b, c and d are ordered as follows: $a < c < b < d$, in which case $(a, b) \cap (c, d) = (c, b)$. But this again is an element of \mathcal{B} . So the second requirement of being a basis is satisfied in the same way as the previous situation. So, \mathcal{B} is a basis for a topology on \mathbb{R} .

Now to show that this topology agrees with the standard topology with basis $\mathcal{S} = \{(a, b) | a < b\}$. We apply the Lemma that Corey screwed up in class last Thursday. Let $(r, s) \in \mathcal{B}$, and let $x \in (r, s)$. Then there exists real numbers a and b so that $r < a < x < b < s$, so that $x \in (a, b) \subseteq (r, s)$, and $(a, b) \in \mathcal{S}$.

Now suppose that $(a, b) \in \mathcal{S}$, and that $x \in (a, b)$. Since \mathbb{Q} is dense in \mathbb{R} , we may find rational numbers r, s so that $a < r < x < s < b$, so that $x \in (r, s) \subseteq (a, b)$, and $(r, s) \in \mathcal{B}$. Thus the hypotheses of the Lemma are satisfied, and the standard basis generates the same topology as the basis \mathcal{B} .

2. Suppose X is a set, and τ is a collection of subsets of X so that (1) $X, \emptyset \in \tau$, (2) for any collection $U_\alpha \in \tau$, $\cup_\alpha U_\alpha \in \tau$, and (3) for any $U_1, U_2 \in \tau$, we have $U_1 \cap U_2 \in \tau$. Prove that τ is a topology on X . (This relaxes the condition of finite intersection to an easier-to-check alternative of checking that the *pairwise* intersection of open sets is open, rather than checking that any *finite* intersection of open sets is open).

Proof. The only hypothesis missing to show that τ is a topology on X is that of intersections. We must show that if U_1, \dots, U_n is a finite collection of open sets, then $U_1 \cap \dots \cap U_n$ is open. Using induction with $n = 2$, we see that this base case is satisfied directly by hypothesis. So now suppose that the intersection of the $n - 1$ open sets $U_1 \cap \dots \cap U_{n-1}$ is open. Then, as the intersection of two open sets,

$$U_1 \cap \dots \cap U_n = (U_1 \cap \dots \cap U_{n-1}) \cap U_n$$

is open.

3. Let X be a set, and let τ be a topology on X . Recall that a *closed* set is of the form $U^c := X - U$ for some open set U . I strongly suggest you use DeMorgan's Laws to prove the following three facts:
 - (a) X and \emptyset are closed.

(b) For any collection F_α of closed sets, then $\bigcap_\alpha F_\alpha$ is closed.

(c) For any finite collection of closed sets F_i ($i = 1, 2, \dots, n$), then $\bigcup_{i=1}^n F_i$ is closed.

Corey seems to recall going through each of these facts in class, and asks that you show mercy on him by referencing your lecture notes and not forcing him to type this out. . . he would be typing out almost exactly what he wrote.

4. Let $X = \mathbb{R}$. Suppose that there is a topology on X for which any set of the form $[a, b)$ and $(a, b]$ are open. Prove that this topology must coincide with the discrete topology (that is, this topology and the discrete topology on X have the same open sets).

Corey also seems to recall going over the solution to this problem in two different ways in class, and hopes again that you won't mind if he doesn't type this out.

5. Make up your own topological space. Be creative! (That is, specify a set X , and a collection of subsets τ , and prove your collection of subsets forms a topological space.)

For this problem, I would encourage each of you to simply go on a literature search. There are A LOT of topologies out there. Lots. Each one created for a different purpose. For those interested, there is a book that Corey's sister got him called Counterexamples in Topology which contains literally hundreds of examples of different topologies, each one constructed with a specific purpose in mind. ROCK ON!