Extra math 555 stuff that Corey seems to want to clarify by typing out for you all

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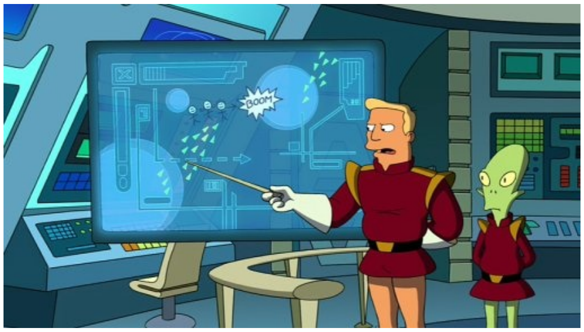

Hello, space cadets! Corey wanted a professional to come in and make sure that you all had seen clean proofs of certain facts, so he called me: Zapp Brannigan! That's right, while not careening through space, I'm hard at work learning topological concepts and Corey did the right thing to call me. It's finally time that his bumbling nincompoopery is put to rest! He wanted me to mention that this document isn't really an extra study guide to the test (for example, there is something on continuity below which will not be on the exam), but the notions contained therein bear a striking resemblance to some of the notions that Corey finds most important. In particular, the notions of continuity aren't as much of interest to you now, but the tools to study these notions really make you understand concepts such as the basis for a topology. So, in response, I told him to shut his mouth and get back to his day job as a bathroom attendant at the Hofbräuhaus. Now Kif, type this out as I speak it…let's do some math! ROCK ON!

1 Topics regarding the standard topology on $\mathbb{R}^n$.

The standard topology on $\mathbb{R}^n$ is defined as the topology with basis

$$\mathcal{B} := \{B_\epsilon(x) | \epsilon > 0, x \in \mathbb{R}^n\}.$$

In addition, there is a distance function $d(x, y) = |x - y|$ on $\mathbb{R}^n$ that we use to define these basis elements as

$$B_\epsilon(x) := \{p \in \mathbb{R}^n | d(p, x) < \epsilon\}.$$

The first result completes the proof that for every point z in the intersection of the basis elements $B_\epsilon(x)$ and $B_\delta(y)$ in the standard topology for $\mathbb{R}^n$, there exists a third basis element $B_\gamma(z) \subseteq B_\epsilon(x) \cap B_\delta(y)$.

Theorem 1.1 *Use the notation we have established. Let* $z \in B_\epsilon(x) \cap B_\delta(y)$. *Then* $B_\gamma(z) \subseteq B_\epsilon(x) \cap B_\delta(y)$ *where* $\gamma = \frac{1}{2}\min\{\epsilon - d(x, z), \delta - d(y, z)\}$.

[Kif here: I'm editing this document. I just wanted to say that Zapp was about ready to take credit for coming up with γ on his own, when in fact, our own Steve Johnson came up with it. I wanted to make sure to give credit to Steve for coming up with γ.]

Proof. Let $p \in B_\gamma(z)$. Then $d(p, z) < \gamma$, and we wish to show that $d(p, x) < \epsilon$, and $d(p, y) < \delta$. We remark that by the definition of γ, we have $d(p, z) < \epsilon - d(x, z)$, and $d(p, z) < \delta - d(y, z)$. Now it follows by the triangle inequality that

$$\begin{aligned} d(p, x) &\leq d(p, z) + d(x, z) \\ &< \epsilon - d(x, z) + d(x, z) = \epsilon. \end{aligned}$$

$$\begin{aligned} d(p, y) &\leq d(p, z) + d(y, z) \\ &< \delta - d(p, z) + d(p, z) = \delta. \end{aligned}$$

□

The next item with regard to the standard topology on $\mathbb{R}^n$ is a fact that has been used several times that I think finally needs to be formalized. We will use this fact in what follows as we study continuity.

Let U be an open subset of $\mathbb{R}^n$ (endowed with the standard topology), and let $x \in U$. By definition of the standard topology, there is a basis element $B_\delta(y)$ for which $x \in B_\delta(y) \subseteq U$, and it is not at all clear by definition that one may choose such a basis element which is centered at x. This next result says that for every point in an open set in $\mathbb{R}^n$, there is an ϵ neighborhood *about that point* that is contained in the open set.

Theorem 1.2 *Let* $x \in U$, *and open subset of* $\mathbb{R}^n$ *(endowed with the standard topology). There exists an* $\epsilon > 0$ *so that* $x \in B_\epsilon(x) \subseteq U$.

Proof. By the definition of the standard topology, there exists a δ and $y \in \mathbb{R}^n$ so that $x \in B_\delta(y)$. Set $\epsilon = \frac{1}{2}(\delta - d(x, y))$. We claim $B_\epsilon(x) \subseteq B_\delta(y)$, and if this holds then the proof will be complete, since in that case

$$x \in B_\epsilon(x) \subseteq B_\delta(y) \subseteq U.$$

But checking our claim is straightforward, and follows along the same lines as in the above proof. Let $p \in B_\epsilon(x)$. Then $d(p,x) < \epsilon < \delta - d(x,y)$. Then we see that

$$\begin{array}{rcl} d(p,y) & \leq & d(p,x) + d(x,y) \\ & < & \delta - d(x,y) + d(x,y) = \delta. \end{array}$$

□

Now finally, our proof equating our old notion of continuity on $\mathbb{R}$ with the new notion of continuity found in Section 18 can be polished using Theorem 1.2. We thank Thomas Schellhous as well for making the proof shorter than the one presented in class.

Theorem 1.3 *Let $f : \mathbb{R} \to \mathbb{R}$. Then the following are equivalent:*

1. *For every open set $V \subseteq \mathbb{R}$, we have $f^{-1}(V)$ is open in $\mathbb{R}$.*
2. *For all $c \in \mathbb{R}$ and $\epsilon > 0$, there exists a δ so that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.*

Proof. Suppose condition (1) is satisfied, and let $c \in \mathbb{R}$ and $\epsilon > 0$ be given. The set $B_\epsilon(f(c))$ is open, and so its preimage $f^{-1}(B_\epsilon(f(c)))$ is open and contains the element c. So, according to Theorem 1.2, there exists a δ so that $B_\delta(c) \subseteq f^{-1}(B_\epsilon(f(c)))$. Thus, for this δ, and every $x \in B_\delta(c)$, we have $f(x) \in B_\epsilon(f(c))$, and this is precisely condition (2).

Now suppose condition (2) is satisfied, and let V be an open subset of $\mathbb{R}$. To show $f^{-1}(V)$ is open, we consider any point $c \in f^{-1}(V)$ and show there exists a δ so that $c \in B_\delta(c) \subseteq f^{-1}(V)$.

So choose any $c \in f^{-1}(V)$. Notice that $f(c) \in V$, a set assumed to be open. So by Theorem 1.2, there exists an ϵ so that $f(c) \in B_\epsilon(f(c)) \subset V$. Then there exists a δ so that if $x \in B_\delta(c)$, then $f(x) \in B_\epsilon(f(c)) \subseteq V$. Thus $c \in B_\delta(c) \subseteq f^{-1}(V)$, and $f^{-1}(V)$ is open. □

2 Regarding limit points in a topological space which is generated by a basis.

Let $A \subseteq X$. By definition, a *limit point* of A is a point x so that for every neighborhood U of x, we have $U \cap (A - \{x\}) \neq \emptyset$. This definition is very similar to the first statement in Theorem 17.5. We denoted the set of all limit points of A as A'.

Look closely at Theorem 17.5, which characterizes points in the closure of a set, $\bar{A}$. There were 2 parts to this theorem: the first characterized these points in general (the statement I was referring to above), and the second part gave a description of them in the presence of a topology which had been generated by a basis. We attempt to characterize limit points in a topology generated by a basis, analogously to the second part of Theorem 17.5 which characterizes points in the closure of a set in a topology generated by a basis.

Theorem 2.1 *Suppose the topology of X is generated by the basis $\mathcal{B}$. Let $A \subseteq X$. Then $x \in A'$ if and only if every basis element $B \in \mathcal{B}$ with $x \in B$ has $B \cap (A - \{x\}) \neq \emptyset$.*

Proof. First we assume that $x \in A'$, and we consider a basis element $B \in \mathcal{B}$ with $x \in B$. B itself is open, and $x \in B$ implies that B itself is a neighborhood of x, and thus by hypothesis, $B \cap (A - \{x\}) \neq \emptyset$.

Now let $x \in X$, and suppose that every basis element which is a neighborhood of x intersects $A - \{x\}$ nontrivially. We wish to prove that *every* neighborhood U of x intersects $A - \{x\}$ nontrivially. We do this by contradiction, and suppose that there exists a neighborhood U of x where $U \cap (A - \{x\}) = \emptyset$. Since the topology is generated by $\mathcal{B}$, there exists a $B \in \mathcal{B}$ so that $x \in B \subseteq U$. But then $B \cap (A - \{x\}) = \emptyset$ since $B \subseteq U$. This is a contradiction to our original assumption. $\square$

One last result justifying the name *limit point.*

Theorem 2.2 *Let $A \subseteq \mathbb{R}$, and let $x \in A'$. There exists a sequence $x_n \in A$ so that (a) $x_n \neq x$ for all n, and (b) $x_n \to x$.*

Proof. Let $U_1 = B_1(x)$. This intersects A at some point x_1 which is not equal to x. Now let $U_2 = B_{1/2}(x)$. This neighborhood, too, intersects A at some point other than x, call it x_2. By continuing this process, we get a sequence x_n with $x_n \neq x$ for all n, also satisfying $x_n \in B_{1/n}(x)$. We must show that the sequence $x_n \to x$. Let $\epsilon > 0$ be given. By the Archimedean principle, there exists an $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. Then for $n \geq N$, we have $x_n \in U_n \subseteq U_N$, and so $|x_n - x| < \epsilon$. $\square$

This concludes all of the good things that Corey wanted you to be clear about before the exam. He also hopes you're rocking out all the time!!!!!!! ROCK ON!