Midterm # 2 Solutions

An Escaped Turkey

November 24, 2007

Hi kids! I managed to get away into the mountains and change my name so that no one will ever bother me again around Thanksgiving time! My first order of business: to grade your exams. I've done that, and in general, they look okay. But there are still some misunderstandings that I hope these solutions will clear up. Oh, and ROCK ON!

1. I'll refer you to the book or notes for these definitions and results.

2. (a) Corey sort of wanted you to use the definition of continuity here, although you could have also used a much shorter proof, modeled after part (c). This is sort of why both parts appeared in the same chunk of questions, since they are sort of related, but different enough to still let you get something out of deciding for yourself which one to try. So we use the $\epsilon - \delta$ definition of continuity, and save the other method as part of the other solution.

   Let $\epsilon > 0$ be given. Then $|f(x) - f(c)| = |(2x^2 - 1) - (2c^2 - 1)| = 2|x - c||x + c|$. So if $c = 0$, then we choose $\delta = \sqrt{\epsilon/2}$. Then, we have

   $$2|x - c||x + c| = 2|x|^2 < 2(\sqrt{\epsilon/2})^2 = \epsilon.$$

   If $c > 0$, then we determine δ as follows. If $0 < \delta < c/2$, then $|x - c| < \delta$ implies

   $$-c/2 < x - c < c/2 \Rightarrow 0 < c/2 < x + c < 3c/2.$$

So this means that in this situation, $|x+c| < 3c/2$, and so

$$2|x-c||x+c| < 2|x-c|\frac{3c}{2} = 3c|x-c|.$$

So if we, in addition, require $0 < \delta < \frac{\epsilon}{3c}$, and $|x-c| < \delta$, then

$$3c|x-c| < 3c\frac{\epsilon}{3c} = \epsilon.$$

So we choose $\delta = \min\{\frac{c}{2}, \frac{\epsilon}{3c}\}$. (We remark that both cases are necessary since if $c = 0$ here, we have δ either undefined, or equal to 0).

(b) Many of you had the right idea, but didn't actually define a value of your function somewhere, making it NOT a function $f : \mathbb{R} \to \mathbb{R}$, but from the real numbers minus a point to the reals. See, a function can't be continuous where it's not defined so anyone who constructed a function whose domain isn't all real numbers would have automatically have constructed a function which is not continuous at those points not in its domain. But this is why Corey stipulated in the problem that your function had to have domain all real numbers. This problem was difficult to grade as a result. Corey wanted to give as much partial credit as possible, but it's tough when the solution involves on object which was not wanted. So scores on this one vary widely, usually because there were some people who (implicitly, mostly) showed that their function wouldn't be continuous no matter HOW you defined it at the points not in its domain. This is a more difficult problem, since it shows that there is NO continuous extension of this function (i.e., there is no function with domain all real numbers that agrees with the old function on the old domain, which is continuous). Corey just wanted you to make up a function and go from there, as I do here:

Let $f(x) = 0$ where $x \neq 0$, and let $f(0) = 5$. We claim this function is not continuous at $x = 0$. Set $\epsilon = 1$. Then suppose there exists a $\delta > 0$ so that $|x| < \delta$ implies $|f(x) - f(0)| < \epsilon$. Then there exists an $x \neq 0$ with $|x| < \delta$ (take, for instance, $x = \delta/2$), and for this x we have $f(x) = 0$. But then $|f(x) - f(0)| = |0 - 5| = 5 \not< \epsilon = 1$.

(c) We prove that every polynomial is continuous by inducting on the degree of the polynomial. Since constant functions are continuous, we have proven the result is true for polynomials of degree 0. Suppose that any polynomial of degree $k-1$ is continuous, and let $p(x)$ be a polynomial of degree k. We may write $p(x) = a_k x^k + q(x)$ where $q(x)$ is a polynomial of degree $k-1$, and $a_k \in \mathbb{R}$. By the induction hypothesis, $q(x)$ is continuous. Since $f(x) = x$ is continuous, so is the repeated product $f(x)^k = x^k$. The polynomial $a_k x^k$ is the product of continuous functions, and hence continuous. Thus we have exhibited $p(x)$ as the sum of continuous functions $a_k x^k$ and $q(x)$, and so $p(x)$ is continuous.

3. (a) Let x_n be a sequence of real numbers. We wish to show there exists a subsequence that has limit in the extended real number system. If x_n is bounded, then by the

Bolzano-Weierstrass Theorem for Sequences there exists a convergent subsequence (with limit in $\mathbb{R}$). If x_n is unbounded, then by a homework problem (depending on whether or not x_n is unbounded above or below) there is a subsequence x_{n_k} with $x_{n_k} > k$ or $x_{n_k} < -k$ for all k. The subsequence has limit either ∞ or $-\infty$.

(b) We show the sequence is Cauchy. Since Cauchy sequences converge, this will suffice. Notice that for $n \geq m$ we have

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\leq r^n + r^{n-1} + \cdots + r^{m+1} \\
&= r^{m+1}(r^{n-m-1} + r^{n-m-2} + \cdots + 1) \\
&= r^{m+1}\left(\tfrac{1-r^{n-m}}{1-r}\right) \qquad (\text{since } r \neq 1) \\
&< r^{m+1}\tfrac{1}{1-r} \qquad (\text{since } 0 < r < 1).
\end{aligned}$$

Let $\epsilon > 0$ be given. Since $0 < r < 1$ the sequence $r^{m+1}\frac{1}{1-r} \to 0$. So there exists an n_0 so that $r^{n_0+1}\frac{1}{1-r} < \epsilon$. Then we have for any $n \geq m \geq n_0$ that $|x_n - x_m| \leq r^{m+1}\frac{1}{1-r} \leq r^{n_0+1}\frac{1}{1-r} < \epsilon$.

(c) (i) Let $x_n = n$. This sequence has limit ∞ by the following argument. Let $M \in \mathbb{R}$ be given (so that (M, ∞) is a neighborhood of infinity). Then there exists an $n_0 > M$. For all $n \geq n_0$, we have $x_n > M$. Hence $x_n \to \infty$.

(ii) Since this sequence has limit infinity, every subsequence of x_n must also have limit infinity. Thus by definition the set of subsequential limits must contain only the element ∞.

(iii) The $\liminf(x_n) = \inf(E)$. Since, by the previous part, $E = \{\infty\}$, we have $\inf(E) = \infty$. So $\liminf(x_n) = \infty$.

4. (a) Let $c \in D$ be an isolated point. Then there exists a neighborhood U of c so that $U \cap D = \{c\}$. Let V be a neighborhood of $f(c)$, in particular, $f(c) \in V$. Then the neighborhood U of c found above satisfies the condition $x \in U \cap D$ implies $f(x) \in V$ since $U \cap D = \{c\}$, and $f(c) \in V$.

   (b) Let $c \in \mathbb{Q}$, and let x_n be a sequence of irrational numbers converging to c (we can do this since both the irrationals and the rationals are dense in $\mathbb{R}$). Then $f(x_n) = 0 \to 0$, but $f(c) = 1$. So we have constructed a sequence x_n which converges to c, but the sequence $f(x_n)$ does not converge to $f(c)$. So f is not continuous at c. If $c \notin \mathbb{Q}$, then we proceed similarly, and let x_n be a sequence of rational numbers converging to c. Then $f(x_n) = 1$, but $f(c) = 0$. So $f(x_n)$ does not converge to $f(c)$ again, and so f is not continuous at c in this case either. Since the real numbers are the union of those numbers which are rational with those which are not, we have considered all possible places where f could be continuous.

(c) Suppose f is continuous at c, and that $f(c) \neq 0$. Then set $\epsilon = |f(c)|/2$, and define the neighborhood $V = (f(c) - \epsilon, f(c) + \epsilon)$. Notice $0 \notin V$. Since f is continuous at c there exists a neighborhood U so that $x \in U$ implies $f(x) \in V$ (in this problem, the domain $D = \mathbb{R}$, so $U \cap D = U$). In other words, for all $x \in U$, we have $|f(x)| \geq \epsilon$.

5. (a) Suppose $x_n \leq y_n$ for all n. Put $\beta = \liminf(y_n)$. Since y_n is bounded, $\beta < \infty$. By a result in the book, there is a subsequence $y_{n_k} \to \beta$. The subsequence x_{n_k} MIGHT NOT CONVERGE (everyone who tried this problem made the mistake of thinking that x_{n_k} converges), but since x_{n_k} is bounded, there exists a convergent subsequence $x_{n_{k_j}} \to x$. Now $y_{n_{k_j}}$ is a subsequence of the convergent y_{n_k}, so it must have limit β as well. Since $x_{n_{k_j}} \leq y_{n_{k_j}}$, we must have $x \leq \beta$. We have exhibited a subsequential limit x of x_n. So as a lower bound of all subsequential limits, the $\liminf x_n \leq x$. Hence
$$\liminf(x_n) \leq x \leq \beta = \liminf(y_n)\,.$$

(b) Let $x_n = \frac{1}{n+1}$. This is a convergent sequence, and so it is Cauchy. Let $f(x) = 1/x$ be defined on $(0, 1)$ (this function is even continuous on (0,1), although the problem didn't ask for this). The sequence $f(x_n) = \frac{1}{1/(n+1)} = n + 1$ is not Cauchy because it doesn't converge (have a limit in $\mathbb{R}$).

(c) Let $x_n \to a \in A$. We wish to prove that $g(f(x_n)) \to g(f(a))$. Since f is continuous at a, we have that the sequence $f(x_n) \to f(a)$. Now, $f(x_n)$ is a sequence in B converging to $f(a)$ and since g is continuous at $f(a)$ we have $g(f(x_n)) \to g(f(a))$.