# Midterm # 1 Solutions

By: The girl from "Spirited Away", and one of her parents

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*Hi kids! I'm a character in a movie that Corey hasn't seen, but still possess the ability to give you some solutions for the midterm. In general, mostly everyone did fine. The average was 77%. I looked at the exams while Corey was playing tennis and include my thoughts below. Be sure to see at the end of this sheet for additional comments I have about how to interpret your score. Please also keep in mind that Corey's comments are all meant to be constructive. In fact, he told me that he wants anyone who has questions or is concerned about their exam to come talk with him. Or me. But he has regular office hours and I apparently have a pig for a father, and don't have regular office hours. If anything, be sure to* ROCK ON!!!

1. (a) Please see Definition 3.4 or Proposition 3.1 on page 40. Literally, word for word. The notion of convergence has been the only thing we have really studied so far, and I suggest very strongly that you memorize this definition.

   (b) Suppose $x_n = \frac{n}{n+2}$. We claim $x_n \to 1$. Let $\epsilon > 0$ be given. Choose $n_0$ so that $\frac{1}{n_0} < \frac{\epsilon}{2}$. Then for all $n \geq n_0$, we have

$$\left|\frac{n}{n+2} - 1\right| = \frac{2}{n+2} \leq \frac{2}{n} \leq \frac{2}{n_0} < \epsilon\,.$$

2. (a) Let $x_1 = 1$. One can see that $x_2 = 5/4$, and so $x_1 \leq x_2$. We inductively assume that $x_k \leq x_{k+1}$ in an effort to show that $x_n$ is monotone increasing. Then

$x_{k+1} = \frac{1}{4}(2x_k + 3) \leq \frac{1}{4}(2x_{k+1} + 3) = x_{x+2}$. We also show that $x_n$ is bounded above by 3/2, again, by induction. Clearly $x_1 = 1 \leq 3/2$, and if $x_k \leq 3/2$ then $x_{k+1} = \frac{1}{4}(2x_k + 3) \leq \frac{1}{4}(2 \cdot (3/2) + 3) = 3/2$. So $x_k$ is monotone increasing and bounded above, hence it converges.

Now it is a triviality to see what $x_n$ converges to, using known limit theorems from Section 3.2. Suppose $x_n \to x$.

$$x = \lim \frac{1}{4}(2x_n + 3) = \frac{1}{4}(2 \lim x_n + 3) = \frac{1}{4}(2x + 3) \Rightarrow x = 3/2\,.$$

(b) First we show that $x_n$ is bounded below by 1 by induction. Clearly $x_1 \geq 1$. Assume $x_k \geq 1$. Then $x_{k+1} = 2 - \frac{1}{x_k} \geq 2 - 1 = 1$. So this sequence is bounded below by 1. In particular, it's not negative.

Let $x_1 = 3$, and $x_{n+1} = 2 - \frac{1}{x_n}$. One can see that $x_2 = 5/3$, and so $x_1 \geq x_2$. We inductively assume that $x_k \geq x_{k+1}$ and now $x_{k+2} = 2 - \frac{1}{x_{k+1}} \leq 2 - \frac{1}{x_k} = x_{k+1}$. So $x_n$ is bounded below and decreasing, which means it converges. Now we can compute what it must converge to. Suppose $x_n \to x$. Notice again that $x_n \geq 1$, and so $\lim x_n \geq 1$ by the squeeze theorem. $\lim \frac{1}{x_n} = \frac{1}{\lim x_n}$.

$$x = \lim x_n = \lim 2 - \frac{1}{x_n} = 2 - \frac{1}{\lim x_n} = 2 - \frac{1}{x}.$$

Multiplying by $x$, and rearranging we have $x^2 = 2x - 1$, or $x = 1$.

*Note: Many of you concluded that the sequence was bounded without showing that it was bounded below or above. If you said what I said about being INCREASING and bounded above (for instance, for part (a)), this is okay, but otherwise you may have missed a point.*

3. (a) See homework # 1 solutions. The converse is false, $x_n = (-1)^n$ works.

   (b) This problem is otherwise stated as "Let $x_n$ be any bounded sequence, not necessarily convergent), and let $y_n \to 0$. Prove $x_n y_n \to 0$." This is also on the first homework assignment's solutions. The converse is false, try $y_n = \frac{1}{n}$ and $x_n = \frac{1}{n^2}$. Then $\frac{y_n}{x_n} = n$, which diverges.

4. (a) Suppose $x_n$ is such a sequence. For all $n$, this sequence satisfies $x_n \in [0, 2]$ for all $n$. Thus, $x_n$ is bounded. By the Bolzano-Weierstrass Theorem, it has a convergent subsequence.

   (b) Suppose $x_n \to x > 0$, and consider $y_n = (-1)^n x_n$. Notice that $y_n$ has two convergent subsequences, namely $y_{2n} = x_{2n}$ and $y_{2n+1} = -x_{2n+1}$. Each of these subsequences are subsequences of the convergent sequence $x_n$ (or its negative), hence they converge, respectively, to $x$ and $-x$. Thus $y_{2n} \to x$ and $y_{2n+1} \to -x$. Since

$x \neq 0$ it follows that $x$ and $-x$ are not equal. So since $y_n$ has two subsequences which converge to different values, $y_n$ must not converge.

5. (a) See the proof of proposition 3.5 on page 58.

   (b) See Theorem 3.1 on page 42.

   (c) See Theorem 3.10 on page 55.

6. Other remarks. Corey will make some other remarks in class specific to some of these problems. But for now he just wants to get it up on the website so you can have it over the weekend. Enjoy! And, ROCK ON!!!!!!!!