

Homework solutions for sections 4.3 and 4.4

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Hello silly humans! This is Jake the cat writing to you from Corey's residence, here to tell you all about your solutions for sections 4.3 and 4.4. Overall, everyone did well, but there were a few negative trends worth pointing out that I thought I would type up. I can see Corey inside spending most of his weekend grading. Poor guy. That's all I see him do anymore. Oh well, ROCK ON!

4.3, # 2 Let $c \in \mathbb{R}$. We will show that f does not have a limit at c . Let x_n be a sequence of rationals converging to c and let y_n be a sequence of irrationals converging to c . Notice that $f(x_n) = 1$ and $f(y_n) = -1$ for all n . Notice, then, that $\lim f(x_n) \neq \lim f(y_n)$. There is a theorem in the book stating that if $\lim_{x \rightarrow c} f(x)$ exists, then the sequences $f(x_n)$ and $f(y_n)$ must both exist and be equal, which we have shown not to be the case. So $\lim_{x \rightarrow c} f(x)$ does not exist at c .

4.3 # 5 *For this problem some people sort of assumed what was wanted to be shown. Yes, the squeeze theorem is a well-known result, and we have proved the result for sequences. We use it below in the proof. But you're asked to prove that it's true for functions. And here is a solution:*

Let $f(x) \leq g(x) \leq h(x)$ for all $x \in D - \{c\}$, and suppose $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$. Let x_n be a sequence in $D - \{c\}$ which converges to c (note that it's not obvious that such a sequence exists. Consider, for example, c is an isolated point. But recall that we have only defined limits at accumulation points, and the assumption that the limits of f and h exist at c implicitly assumes that c is an accumulation point of D . By a previous result discussed in class we know that such a sequence exists). Then the sequence $g(x_n)$ satisfies $f(x_n) \leq g(x_n) \leq h(x_n)$ for all n . By the theorem in this section we know that $\lim f(x_n) = L$ and $\lim h(x_n) = L$. Thus $\lim g(x_n)$ exists, and equals L as well.

4.4 #4 Since f is a polynomial, it is continuous on all of \mathbb{R} , and so in particular, it is continuous on $[0, 2]$. Notice that $f(0) > 0$, $f(1) < 0$ and $f(2) > 0$. So by the Intermediate Value Theorem, since $f(1) < 0 < f(0)$, there exists a $c_1 \in (0, 1)$ so that $f(c_1) = 0$. Similarly, since $f(1) < 0 < f(2)$ there exists a c_2 so that $f(c_2) = 0$. Thus f has (at least) 2 roots in $[0, 2]$.

4.4 # 7 *This problem was generally pretty good, most people proved this by contradiction. But there were a few people who assumed they knew the extent to which f was not constant, which is something which can't be assumed. So here is a solution that would avoid that.*

Suppose not, that there exist $c < d \in (a, b)$ with $f(c) \neq f(d)$. Since f is continuous on (a, b) it is continuous on $[c, d]$, and so the intermediate value theorem applies on $[c, d]$. Since $f(c) \neq f(d)$ there exists a k strictly between $f(c)$ and $f(d)$ which is not an integer. But then, by the IVT, there must exist a $p \in (c, d)$ with $f(p) = k \notin \mathbb{Z}$, which is a contradiction. So every continuous integer-valued function on a connected interval must be constant.