HW solutions for Sections 3.8, 4.1 and 4.2

By: Finals Week

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Well, I'm coming soon. Finals week. And so here are the solutions to the homework from sections 3.8, 4.1 and 4.2. Nobody likes finals week. Why? ROCK ON!

3.8, # 1 These limsup's and liminf's are not hard to compute. Just remember, for part (f) that $\lim e^{-n} = 0$, and since the limit exists, we have $\limsup e^{-n} = \liminf e^{-n} = \lim e^{-n} = 0$.

(a) $\limsup x_n = 1, \liminf x_n = -1$

(b) $\limsup x_n = \infty, \liminf x_n = -\infty$

(c) $\limsup x_n = 1, \limsup x_n = -1$

(f) $\limsup x_n = \liminf x_n = 0$.

4.1, # 2 Let $c > 0$, and let $\epsilon > 0$ be given. Notice that for $x > 0$ we have

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{xc}.$$

If $\delta \leq c/2$, then we have $|x - c| < c/2$ and hence $-c/2 < x - c < c/2$ and so $c/2 < x$. Then $\frac{|x-c|}{xc} \leq |x - c| \cdot \frac{2}{c^2}$. So we set $\delta = \min\{c/2, \epsilon c^2/2\}$. Then we have

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{xc} \leq |x-c| \cdot \frac{2}{c^2} \leq \epsilon \frac{c^2}{2} \frac{2}{c^2} = \epsilon.$$

4.1, # 5 Set $\epsilon = 1$. Then since f is continuous at c, there exists a neighborhood U so that $|f(x) - f(c)| < 1$ for all $x \in U \cap D$. But then

$$|f(x) - f(c)| < 1 \Rightarrow ||f(x)| - |f(c)|| < 1 \Rightarrow -1 < |f(x)| - |f(c)| < 1,$$

so for all $x \in U \cap D$, we have $|f(x)| < 1 + |f(c)|$. Thus f is bounded on $U \cap D$.

4.2, # 6 We are given that $f(x) = c$ for all $x \in \mathbb{Q}$. So let $\zeta \in \mathbb{R} - \mathbb{Q}$, and let x_n be a sequence of rational numbers converging to ζ. Since it is given that f is continuous, we know that $f(x_n) \to f(\zeta)$. But since $x_n \in \mathbb{Q}$ for all n, we have $f(x_n) = c$ for all n. This sequence converges to c. By the uniqueness of limits, we conclude $f(\zeta) = c$.