HW #2 Solutions

By: The people that put you on hold at the phone company

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Corey was able to type almost this entire document while on hold! Isn't that a wonderful use of his time?! Here are the solutions to the second homework assignment. Rock on!

1. (a) Let x_n be the sequence 1,0,2,0,3,0, ... Then x_n is clearly unbounded, and the sequence x_{2n}, being the constant sequence, converges.

   (b) Let $x_n = n$. Clearly x_n is unbounded, and we show that there does not exist a convergent subsequence. Suppose $x_{n_k} = n_k > k$ is a subsequence of x_n. Then under the assumption that x_{n_k} converges, it must be bounded. But $x_{n_k} = n_k \geq k$ for all k. So this subsequence is unbounded, which is a contradiction.

   (c) The Bolzano-Weierstrass Theorem says that we may not find a bounded sequence with no convergent subsequence.

2. We discussed these in class. Namely, $(1+\frac{1}{n^2})^{n^2}$ is a subsequence of the sequence $(1+\frac{1}{n})^n$, which converges to e. So $(1+\frac{1}{n^2})^{n^2}$ converges to e as well. For part (b) notice that $(1+\frac{1}{2n})^n$ is the square root of the convergent subsequence $(1+\frac{1}{2n})^{2n}$. So the sequence $(1+\frac{1}{2n})^n$ must converge to $\sqrt{e}$.

4. Let z_n be the sequence $(x_1, y_1, x_2, y_2, \ldots)$. Show that z_n converges if and only if x_n and y_n converge to the same limit.

   Proof. If z_n converges, then, as subsequences of z_n, we have x_n and y_n are convergent and converge to the same limit.

   Conversely, if $x_n, y_n \to z$, then we must show z_n converges to z. Let $\epsilon > 0$ be given. Choose n_1 so that $|x_n - z| < \epsilon$ for all $n \geq n_1$. Then notice that for all $n \geq 2n_1 - 1$, we have $|x_{(n+1)/2} - z| < \epsilon$. Similarly, we choose n_2 so that for all $n \geq n_2$ we have $|y_n - z| < \epsilon$. Notice that for all $n \geq 2n_2$, we have $|y_{n/2} - z| < \epsilon$. Notice

   $$z_n = \begin{cases} x_{(n+1)/2} & n: \text{ odd} \\ y_{n/2} & n: \text{ even} \end{cases}.$$

   Choose $n_0 = \max\{2n_1 - 1, 2n_2\}$. Then we have if $n \geq n_0$ and is odd, then $|z_n - z| < \epsilon$, and if $n \geq n_0$ is even, we have $|z_n - z| < \epsilon$ as well. So $z_n \to z$.

5. (a) Let x_n be an unbounded sequence. There exists an n_1 so that $x_{n_1} > 1$. Let $n_2 > n_1$ be an index with $x_{n_2} > 2$. We argue that this may be done, and conclude that we may continue this process indefinitely to obtain the sequence $x_{n_k} > k$. Suppose there is no index n_2 with $x_{n_2} > 2$. Then for all $n > n_1$ we have $x_n \leq \max\{2, x_1, \ldots, x_{n_1}\}$, contradicting the hypothesis that x_n is unbounded from above.

   (b) Set $y_n = -x_n$. Then since x_n is unbounded from below, we have y_n is unbounded from above. So there exists a subsequence of y_n with $y_{n_k} > k$. But then since $y_{n_k} = -x_{n_k} > k$, we have $x_{n_k} < -k$.

Rock on!!!!