

# HW #1 Solutions and Comments

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*Hello, everyone, this is Fry. I've been living in the year 3000, and as you can see, Starbucks still has a stranglehold on society. Corey finished grading your homeworks and decided that he would try to compromise with everyone and type up the homework questions he graded, and provide any other comments on ones that he didn't that warrant some kind of commentary. Actually, he's too busy for even that, so he'll often pass that chore down to some friends of his, like me. They won't be his friends for long, though, if he keeps that up! Here are some solutions and comments about the homework. ROCK ON!*

- 3.1, #2. Let  $\epsilon > 0$  be given. Choose  $n_0$  so that  $1/n_0 < \epsilon$ . Then we claim that  $|x_n - 0| < \epsilon$  for all  $n \geq n_0$ . If  $n$  is odd, then  $x_n = 1/n$ , and  $|1/n - 0| = |1/n| \leq 1/n_0 < \epsilon$ . If  $n$  is even, then  $x_n = 0$ , and clearly  $|x_n - 0| = 0 < \epsilon$ . So for all  $n \geq n_0$ , we have  $|x_n - 0| < \epsilon$ .

*Note: Most people tried to simply state that  $1/n \rightarrow 0$  and  $0 \rightarrow 0$ , and that it is for this reason that  $x_n \rightarrow 0$ . But what you've really done is just find two subsequences which converge to 0. Consider the sequence  $x_n = \sin\left(\frac{n\pi}{2}\right)$ . The numbers  $n$  which are divisible by 4 and the  $n$  which are 2 greater than a number divisible by 4 both give you (distinct) subsequences which converge to 0. But as you all know, this sequence doesn't converge. An  $\epsilon$  proof is needed for this one, and it has to be done carefully.*

- 3.1, #4. (a) Let  $\epsilon > 0$  be given. Since  $x_n \rightarrow x$ , we may choose  $n_0$  so that  $|x_n - x| < \epsilon$ . Then for all  $n \geq n_0$ , we have

$$||x_n| - |x|| \leq |x_n - x| < \epsilon.$$

*Note: Many of you seemed to assume what you were trying to prove, but many of you keyed in on the main ideas as well. Just make sure to find the  $n_0$ , and say things like “for all  $n \geq n_0, \dots$ ”.*

(b) This one I believe I did in class. Most all of you got full credit here.

(c) Try  $x_n = (-1)^n$ .

3.2, #2. The sequences  $x_n = (-1)^n$  and  $y_n = (-1)^{n+1}$  will do for all parts.

3.2, #3. Suppose  $x_n + y_n$  converge, and that  $x_n - y_n$  converge. We show that each of  $x_n$  and  $y_n$  converge as well. Notice that for each  $n$ , we have

$$x_n = \frac{(x_n + y_n)}{2} + \frac{(x_n - y_n)}{2}.$$

Since  $x_n + y_n$  and  $x_n - y_n$  both are convergent sequences, it follows (from a theorem in the book) that  $\frac{(x_n + y_n)}{2}$  and  $\frac{(x_n - y_n)}{2}$  each converge. Since each of these sequences converge, their sum must converge. Since, for each  $n$ , the sequence  $x_n$  is this sum, we conclude  $x_n$  converges. The same line of reasoning applies to  $y_n$  once you note that

$$y_n = \frac{(x_n + y_n)}{2} - \frac{(x_n - y_n)}{2}.$$

*Note: A lot of you tried an  $\epsilon$  argument here, but made the mistake of concluding that  $|a| + |b| = |a + b|$ , or worse, that  $|a| - |b| = |a - b|$ . In addition, there were implicit statements that some of you assumed that you knew what  $x_n$  and  $y_n$  should converge to by stating that  $x_n + y_n \rightarrow x + y$ . All you really know is that  $x_n + y_n$  converges to some number, say  $A$ , and it is suggestive (but not technically incorrect) to suggest that it converges to  $x + y$ . So I point this out in passing, and of course, didn't mark anyone off for this. Only a few people didn't somehow suggest this in their work. The real key to this problem is to notice that the hypotheses allow you to express both  $x_n$  and  $y_n$  as the sum or difference of convergent sequences.*

3.2, #6. Since  $x_n$  is bounded, there exists a  $B \in \mathbb{R}$  so that  $|x_n| \leq B$  for all  $n$ . We may choose  $B > 0$ , since any number greater than  $B$  is also a bound for the sequence  $x_n$ . In particular, if  $B = 0$ , then any  $B > 0$  will also bound the sequence. Let  $\epsilon > 0$  be given. Since  $y_n \rightarrow 0$  we may choose  $n_0$  so that  $|y_n| < \epsilon/B$  for all  $n \geq n_0$ . Then

$$|x_n y_n - 0| = |x_n| |y_n| < B \cdot \epsilon/B = \epsilon.$$

*Note: Almost all of you didn't address the possibility that  $B = 0$  when you chose  $n_0$  so that  $|y_n| < \epsilon/B$ . This is a minor quibble, but certainly justified. I've done the proof above purposely to address this issue, although one could have just chosen  $n_0$  so that  $|y_n| < \epsilon/(B + 1)$  as well. Convince yourself that this would work as well.*

Others. Although Corey didn't grade 3.1 #3, he seems to think that many of you concluded that  $x_n$  did NOT converge because of the following reason: because there exist subsequences which converge to two different limits. Corey didn't want to grade this one since it arguable that you're not supposed to know this in Section 3.1. We cover that theorem in Section 3.3 (Theorem 3.6). The argument, technically, is correct... just a little ahead of its time. What I think is wanted for that one is for you to try to prove this sequence doesn't converge by showing that the definition of convergence doesn't hold, not by any more advanced means. So I suggest that, for those of you who didn't do that, that this may be an excellent exercise since it asks you to understand what negating this definition is all about. That is, to learn it "backwards and forwards", so to speak.

Other, more random, comments include stylistic commentary. In particular, although Corey can sort of tell what you're saying when you write a bunch of symbols down, do try to be careful. For instance, several of you used a squeeze theorem argument for 3.2 #6 that ended with

$$0 \leq |x_n y_n| \leq B|y_n| \rightarrow 0.$$

This argument sort of leaves a few things to be desired which deserve their own commentary. For instance, the actual conclusion here is that the sequence  $|x_n y_n| \rightarrow 0$ , not  $x_n y_n \rightarrow 0$ . Yes, we have a result (3.1, #4) which says that if we could show  $|x_n y_n| \rightarrow 0$  we would be able to conclude that  $x_n y_n \rightarrow 0$ . But don't you think this needs to at least be mentioned? Also, conversely, you're only told that  $y_n \rightarrow 0$ , and one uses the fact that  $|y_n| \rightarrow 0$  to establish the squeezing above. Yes, we have a result (3.1, #4 again) which says this is true, but again, there's a few skipped steps.

Maybe now a lot of you are starting to realize that the problems I have you do are not as random as they seem. Whenever possible Corey assigns problems which go together, or will be of importance later. The fact that  $|x_n| \rightarrow 0 \Rightarrow x_n \rightarrow 0$  is a very important fact that comes up all the time. This doesn't mean that you should freak out all the time about homework, but it does mean that Corey has attempted to organize this class in a very thoughtful way because he wants you all to succeed. He knows that this course can cause a lot of anxiety, and is doing his best on his end to make it as smooth as possible for you. Stay tuned in class, we will be discussing this further there as well. But for now, ROCK ON!