Name: ________________________________

STOLEN Math 546 Final Exam of Goodness!!!!

Directions: Please take this exam without cheating. It's worth 200 points. You may use a calculator, a writing utensil, and your brain. Good luck!

1. (25 points) Please do exactly TWO of the following to do. No, you won't get extra credit if you do more.

   (a) Let R be a set with binary operations $+$ and $\cdot$ ....

   (b) Let R be a ring. Please carefully define what it means for R to be an .... Please also carefully state the ....

   (c) Let R be a ring, and $I \subseteq R$. Please carefully define what it means for I to be .... If $a \in R$ and R is a commutative unital ring, ... $< a >$, the........................ and prove ....

   (d) Let R be a ring, and let I be an ideal of R. ....

   (e) Let R be a ring, and let I be an ideal of R. ....

2. (25 points) Please do exactly TWO of the following to do. No, you won't get extra credit if you do more.

   (a) Let F be a field, and suppose $f \in F[x]$. Please carefully define what it means for f to be........................ Please also give an example of a field F, and a polynomial........................ irreducible.

   (b) Let $\varphi : R \to R'$ be a function, and let R and R' be rings. ....

   (c) .... the *Division Algorithm* as it applies to ....

   (d) ....

3. (25 points) Please do exactly ONE of the following to do. No, you won't get extra credit if you do more.

   (a) Let R be a ring, and let A be a subring of R, and let $\varphi : R \to R'$ be a ring homomorphism. Please do all of the following:

      i. Show that $\varphi(A)$ is ....

      ii. Please show that $\ker \varphi = \{0\}$ if and only if ....

   (b) Let R be a ring with multiplicative identity $1 \in R$, and let I be an ideal of R. Please do all of the following:

      i. ....

      ii. Please show that if R is a field, then $I = \{0\}$ or $I = R$.

   (c) .... show that for all $x, y \in R$ that ....

   (d) Let R be a ring with additive identity 0. Please do all of the following:

      i. Please show that $0 \cdot a = 0$ for all $a \in R$.

      ii. Please show that cancellation ....

      iii. Please give an example of ....

4. (25 points) Please do exactly ONE of the following to do. No, you won't get extra credit if you do more.

   (a) Let R, R' be rings, and let $\varphi : R \to R'$ be a ring homomorphism. Please do all of the following:

      i. .... $\varphi(I)$.........................

ii. Please draw a smileyface on the upper right-hand corner of the page after you read question 3.b.ii on this exam. Now suppose that R' is a field, and that φ is onto. Please show that ....

iii. Please give an example of rings R, R', a ring homomorphism $\varphi : R \to R'$, and I an ideal of R where (a) R' is a field, (b) I is a nontrivial ideal of R, and (c) .... Demonstrate that $\varphi(I)$ is an ideal of $\varphi(R)$, and that it is not necessarily the case that $\varphi(I)$ is ....

(b) Let I be an ideal of a ring R. Please do all of the following:

   i. Please show that if I is maximal, then ....
   ii. Give an example of a ring R and an ideal I where I is prime, but not maximal.

(c) Let $f \in F[x]$, where F is a field. Please do all of the following:

   i. Let $a \in F$. Please prove that a is a root of f if and only if $x - a$ is a factor.
   ii. Suppose $deg(f)$ is 2 or 3. Please prove that f is irreducible if and only if ....
   iii. Please give an example of a field F and a reducible polynomial of degree 4 that .... (This demonstrates the importance of the hypothesis in........................)

(d) Let R be a ring. Please do all of the following:

   i. Please carefully define ....
   ii. Show that if $R \cong R'$, then ....
   iii. Please show that ...., the least common multiple of m and n.
   iv. Please use the previous question to show that........................

5. (25 points) Please do exactly ONE of the following to do. No, you won't get extra credit if you do more.

(a) Let $f, g, h \in F[x]$, where F is a field. Please do all of the following:

   i. Suppose f is irreducible. Please show that if $f|gh$, then ....
   ii. Give an example of .... but f does not divide g or h.

(b) Let D be an integral domain. Please do all of the following:

   i. ....$D[x]$ .... integral domain.
   ii. Given........................
   iii. Show that the units of $D[x]$ are precisely the units of D.

(c) Please carefully state and prove .... for determining if a polynomial $f \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$. Please also .... criterion ....degree 7........................

(d) Let D be an integral domain. Please do all of the following:

   i. Let I be an ideal of D. Define the *radical of I in D* to be $\sqrt{I} = \{x \in D | x^n \in I \text{ for some } n \in \mathbb{Z}\}$. Show that ....
   ii. Prove that $\{0\}$ is .... prime ideal of ....
   iii. Please show that $\sqrt{\{0\}} = \{0\}$ in our integral domain D.
   iv. Give an example of a commutative ring R in which ....

6. (25 points) Please do exactly TWO of the following to do. No, you won't get extra credit if you do more.

(a) Please determine if the rings .... and .... are isomorphic. If so, please produce an isomorphism. If not, then please clearly demonstrate why.

(b) Please determine if the rings .... and .... are isomorphic. If so, please produce an isomorphism. If not, then please clearly demonstrate why.

(c) Please determine if the rings $\mathbb{Q}[x]/\langle x - 8 \rangle$ and $\mathbb{Q}$ are isomorphic. If so, please produce an isomorphism. If not, then please clearly demonstrate why. Corey urges you to consider the remainder theorem for this one.

(d) Please determine if the rings .... and .... are isomorphic. If so, please produce an isomorphism. If not, then please clearly demonstrate why.

7. (25 points) Please do exactly ONE of the following to do. No, you won't get extra credit if you do more.

   (a) Suppose $f \in \mathbb{Q}[x]$. Please do all of the following:

      i. Please remind yourself that if $f(x)$ is a polynomial, then $f(x+1)$ is a polynomial. Please then try not to become afraid that we could study the irreducibility of $f(x+1)$, just like we could study the irreducibility of $f(x)$.
      ii. Show that $f(x)$ is irreducible ....
      iii. Please show that if p is prime, then the polynomial $\Phi_p(x) =$ .... is irreducible over $\mathbb{Q}$.

   (b) Suppose F is a field. Please do all of the following:

      i. .... principle ideal domain.
      ii. Please show that $\mathbb{Z}[x]$ is *not* ....

   (c) ....

   (d) ....

8. (25 points) Please do exactly ONE of the following to do. No, you won't get extra credit if you do more.

   (a) Please do all of the following:

      i. ....
      ii. Please find all of the zeros of the polynomial........................ Why does this not ....?

   (b) Please do all of the following:

      i. Construct a field having ....
      ii. What is the characteristic of the field you constructed above?
      iii. Is it true that *every* field having .... has characteristic 3? (Please remember that there are fields........................ were not constructed from polynomial rings.)

   (c) Please do all of the following parts:

      i. Let $f(x) =$ ...., and $g(x) =$ ...., with $f, g \in \mathbb{Z}_2[x]$. Please compute........................
      ii. Please find polynomials $u, v \in \mathbb{Z}_2[x]$ with $f(x) \cdot u(x) + g(x) \cdot v(x) = 1$.
      iii. Please compute the multiplicative inverse of $f(x)$ in $\mathbb{Z}_2[x]$.

   (d) Please do all of the following:

      i. .... uniqueness of factoring .... $f \in F[x]$.
      ii. Suppose $f(x) =$ .... Please completely factor $f(x)$ according to........................ (Hint: $x = \frac{1}{2}$ is a root of $f(x)$.)

9. For 2 free points, please tell me something funny!!!