Notes about the Quaternions, and the Subfield test

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May 11, 2009

Hi kids! This document is really just about two different and somewhat unrelated aspects of Section 6.3: the quaternions, and the subfield test. Corey asked me to please write a handout for all of you since he's so busy with everything else. After all, I'm one of his mentors, so I decided it might be fun. ROCK ON!

1 The Subfield Test.

We give a good proof of this test, Corey really isn't that happy with how it went in class.

Theorem. (Subfield Test) Suppose $\emptyset \neq S \subseteq F$. Then S is a subfield of F if and only if for all $a, b \in S$, we have (1) $a - b \in S$, and (2) for $b \neq 0$, we have $ab^{-1} \in S$.

Proof. It's more or less obvious that if S is a subfield of F, then (1) (S is a subring) and (2) (inverses and products are closed in fields) hold. So conversely, suppose (1) and (2) hold. We have to show S is a subring of F, S is commutative, $1 \in S$, and that every nonzero element of S has a multiplicative inverse. First, we choose $a = b$ in (2) to conclude that $aa^{-1} = 1 \, inS$. Now we use $a = 1 \in S$, and any $0 \neq b \in S$ to conclude $1 \cdot b^{-1} = b^{-1} \in S$. Now we use $a \in S$, and $0 \neq b^{-1} \in S$ to conclude $a(b^{-1})^{-1} = ab \in S$. If $b = 0$, then it is clear that $ab = 0 \in S$, since, by (1) with $a = b$, we have $a - b = 0 \in S$. By the subring test, S is a subring of F. Since $S \subset F$, a commutative ring, all elements of S commute. So S is a subfield of F.

2 Hamilton's Quaternions.

We discussed this ring in detail in class. Corey vowed to give you all a better proof as to why every nonzero element has an inverse. It will follow that there are no zero divisors in $\mathbb{H}$. I refer you to Example 6.3.20 on page 206 for a definition of $\mathbb{H}$. So here it goes. Define

$$\Theta := \{x_1 i + x_2 j + x_3 k | x_i \in \mathbb{R}\}.$$

Now define the function $||\cdot|| : \mathbb{H} \to \mathbb{R}$ as $||x_0 + x_1 i + x_2 j + x_3 k|| = \sqrt{\sum x_i^2}$. It is clear that $||\alpha|| = 0$ if and only if $\alpha = 0$. Now suppose $\alpha = x_0 + \theta \neq 0$. We claim that

$$\alpha^{-1} = \frac{x_0 - \theta}{||\alpha||^2}.$$

We check this by first supposing that $\theta = x_1 i + x_2 j + x_3 k$, and notice two facts before we begin. For any $x_0 \in \mathbb{R}$, we have

$$x_0\theta = x_0(x_1 i + x_2 j + x_3 k) = x_0 x_1 i + x_0 x_2 j + x_0 x_3 k = x_1 i x_0 + x_2 j x_0 + x_3 k x_0 = \theta x_0.$$

Namely, real scalars commute with Θ. Also, we compute

$$\begin{aligned}
\theta^2 &= (x_1 i + x_2 j + x_3 k)(x_1 i + x_2 j + x_3 k) \\
&= -x_1^2 + x_1 x_2 ij + x_1 x_3 ik + x_2 x_1 ji - x_2^2 + x_2 x_3 jk + x_1 x_3 ki + x_3 x_2 kj - x_3^2 \\
&= -x_1^2 - x_2^2 - x_3^2 + x_1 x_2 k - x_1 x_3 j - x_2 x_1 k + x_2 x_3 i + x_1 x_3 j - x_3 x_2 i \\
&= -x_1^2 - x_2^2 - x_3^2.
\end{aligned}$$

Now we are ready to prove our claim. Notice that

$$\begin{aligned}
\alpha\alpha^{-1} &= (x_0 + \theta)\tfrac{(x_0 - \theta)}{||\alpha||^2} \\
&= \tfrac{1}{||\alpha||^2}(x_0 + \theta)(x_0 - \theta) \\
&= \tfrac{1}{||\alpha||^2}(x_0^2 - x_0\theta + \theta x_0 - \theta^2) \\
&= \tfrac{1}{||\alpha||^2}(x_0^2 - \theta^2) \\
&= \tfrac{1}{||\alpha||^2}(x_0^2 + x_1^2 + x_2^2 + x_3^2) \\
&= 1.
\end{aligned}$$

Rock on!!!