

Midterm Solutions

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Hello, fellow law-abiding citizens of Corey's Math 546 class. I couldn't help but spy on you all while you were taking the exam, and I've solved all of the problems and present to you all my findings. Don't do drugs! ROCK ON!

1. These I think you'll find in the book.
2. (a) Suppose cancellation holds in R , and that there exists $a, b \in R$ so that $ab = 0$. Note $ab = 0 = a \cdot 0$ and since cancellation holds, $b = 0$, so there must not be any zero divisors. Conversely, suppose there are no zero divisors in R and $a, b, c \in R$, $a \neq 0$ with $ab = ac$. Then $ab - ac = a(b - c) = 0$ and thus, since there are no zero divisors and $a \neq 0$, we must have $b - c = 0$, or $b = c$. A ring where cancellation does NOT hold is \mathbb{Z}_{12} . Note $4 \cdot 3 = 4 \cdot 9 = 0 \pmod{12}$, but that $3 \neq 9 \pmod{12}$.
(b) If I is maximal, then R/I is a field. Any field is also an integral domain, and hence R/I is an integral domain, and it follows by a previous result that I must be prime.
(c) This is proved on a previous homework writeup.
(d) Any ring homomorphism $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}$ must also be a group homomorphism, and hence is determined by what $\varphi(1)$ is. Suppose $\varphi(1) = a \in \mathbb{Z}$. Since φ is a ring homomorphism, we must have $a = \varphi(1) = \varphi(1 \cdot 1) = \varphi(1)\varphi(1) = a^2$. Since \mathbb{Z} is an integral domain, we must have $a = 0$ or 1 , and so the only two ring homomorphisms are the zero homomorphism and the identity.

3. (a) (i) $a \in R$ is defined to be *idempotent* if $a^2 = a$.
- (ii) Suppose a is idempotent. Then $\varphi(a)^2 = \varphi(a^2) = \varphi(a)$.
- (iii) Let a be idempotent in an integral domain. Then $a^2 = a$, or $a^2 - a = a(a-1) = 0$. Since R is an integral domain, this implies $a = 0$ or $a - 1 = 0$, in any case, $a = 0$ or 1 . One checks that both of these elements in fact are idempotent.
- (b) (i) We use the subring test. Let $\varphi(a), \varphi(b) \in \varphi(R)$. Then first we note $\varphi(a) - \varphi(b) = \varphi(a - b) \in \varphi(R)$, and second $\varphi(a)\varphi(b) = \varphi(ab) \in \varphi(R)$. So by the subring test, $\varphi(R)$ is a subring of R' .
- (ii) Let $\varphi(b) \in \varphi(R)$. Then $\varphi(1)\varphi(b) = \varphi(1 \cdot b) = \varphi(b) = \varphi(b \cdot 1) = \varphi(b)\varphi(1)$. Hence $\varphi(1)$ is the unit in $\varphi(R)$.
- (iii) Let $R = \mathbb{Z}$ and let $R' = \mathbb{Z} \times 2\mathbb{Z}$. Define the homomorphism $\varphi : R \rightarrow R'$ as $\varphi(n) = (n, 0)$. This is clearly a homomorphism, and the image of the multiplicative identity $1 \in R$ is the element $(1, 0) \in \mathbb{Z} \times 2\mathbb{Z}$. This is the multiplicative identity in $\varphi(R) = \{(n, 0) | n \in \mathbb{Z}\}$, although the ring R' has no multiplicative identity. Can you hear me mumbling that?
- (c) This is a homework question already typed up.
- (d) (i) Suppose $1 \in I$. Then since I is an ideal, the elements $r \cdot 1 \in I$ for all $r \in R$, and hence $R \subseteq I$. Therefore, $R = I$.
- (ii) Suppose R is a field, and that I is a nontrivial ideal of R . Then there exists an element $0 \neq a \in I$. Since R is a field, there exists $a^{-1} \in R$. Since I is an ideal, the element $a^{-1}a = 1 \in I$. By the previous part, this implies $I = R$.
4. (a) This is carefully stated and proved in the book.
- (b) Suppose R/I is finite. Since I is prime, R/I is an integral domain. Any finite integral domain is a field, hence R/I is a field, and by a previous result, I is maximal. I lack the LaTeX abilities to draw a smiley face on the top, right-hand corner of the exam.
- (c) This is Theorem 6.3.14 in the book, it's proved there.
- (d) (i) and (ii) are definitions in the book, and (iii) is Theorem 6.3.5 in the book. As for (iv), the integers are a ring that contains the element 2, which is neither a zero divisor (\mathbb{Z} is an integral domain, and has NO zero divisors), or a unit (since $\frac{1}{2}$ is not an integer). (v) Any field is an example, since every nonzero element is a unit. Any \mathbb{Z}_n , where n is composite, is a ring where every nonzero element is a unit or a zero divisor, and that there are some of each.