

Homework # 4, Section 7.1

The rematch between Peter Griffin and the giant Chicken

May 11, 2009



Section 7.1: 1, 3, 11, 12, 15–20, 23.

1. I feel as though we discussed this at length in class, so I'll skip it here.

3. We know that any group homomorphism $\varphi : (\mathbb{Z}_4, +) \rightarrow (\mathbb{Z}_6, +)$ is determined by where it sends the generator 1 of \mathbb{Z}_4 . The map will not be well defined if $\varphi(1) = 1, 2, 4,$ or 5 . Thus we conclude that $\varphi(1) = 3$ and $\varphi(1) = 0$ are the only two possibilities. $\varphi(1) = 0$ clearly defines a group homomorphism, while $\varphi(1) = 3$ defines a group homomorphism $\varphi(n) = 3n \in \mathbb{Z}_6$. Checking that it preserves multiplication: $\varphi(mn) = 3mn$, whereas $\varphi(m)\varphi(n) = 3m3n = 9mn = 3mn$ in \mathbb{Z}_6 since 3 is idempotent there. So there are 2 ring homomorphisms between $\mathbb{Z}_4 \rightarrow \mathbb{Z}_6$.

11. We discussed this one pretty thoroughly in class... we counted 9 possible homomorphisms.

12. We assume that $\varphi : R \rightarrow R'$ and $\psi : R' \rightarrow R''$ are ring isomorphisms. (1) We prove that $\psi \circ \varphi : R \rightarrow R''$ is a ring isomorphism. First note that since φ and ψ are isomorphisms, they are bijective, and so as the composition of bijections, $\psi \circ \varphi$ is a bijection. Now note that since each of ψ and φ are homomorphisms, for $a, b \in R$, we have

$$(\psi \circ \varphi)(a + b) = \psi(\varphi(a) + \varphi(b)) = \psi(\varphi(a)) + \psi(\varphi(b)) = (\psi \circ \varphi)(a) + (\psi \circ \varphi)(b),$$

and

$$(\psi \circ \varphi)(ab) = \psi(\varphi(a)\varphi(b)) = \psi(\varphi(a))\psi(\varphi(b)) = (\psi \circ \varphi)(a)(\psi \circ \varphi)(b).$$

So $\psi \circ \varphi$ is a ring homomorphism, and since we already showed it is bijective, it is a ring isomorphism.

(2) The identity map $i : R \rightarrow R$ is clearly bijective, and easily satisfies the ring homomorphism axioms, hence, it is an isomorphism:

$$i(a + b) = a + b = i(a) + i(b), \quad i(ab) = ab = i(a)i(b).$$

(3) Consider $\varphi^{-1} : R' \rightarrow R$. The inverse of a bijection is a bijection. Now we note that $\varphi(\varphi^{-1}(a + b)) = a + b$, and

$$\varphi(\varphi^{-1}(a) + \varphi^{-1}(b)) = \varphi(\varphi^{-1}(a)) + \varphi(\varphi^{-1}(b)) = a + b.$$

Since φ is a bijection, we conclude that $\varphi^{-1}(a+b) = \varphi^{-1}(a) + \varphi^{-1}(b)$. Similarly, $\varphi(\varphi^{-1}(ab)) = ab$, and

$$\varphi(\varphi^{-1}(a)\varphi^{-1}(b)) = \varphi(\varphi^{-1}(a))\varphi(\varphi^{-1}(b)) = ab.$$

Since φ is a bijection, we conclude that $\varphi^{-1}(ab) = \varphi^{-1}(a)\varphi^{-1}(b)$. So φ^{-1} is a ring isomorphism.

15. Define $\phi(a + b\sqrt{3}) = \begin{bmatrix} a & 3b \\ b & a \end{bmatrix}$. We check that ϕ is a ring homomorphism:

$$\begin{aligned} \phi((a + b\sqrt{3}) + (c + d\sqrt{3})) &= \phi((a + c) + (b + d)\sqrt{3}) \\ &= \begin{bmatrix} a + c & 3(b + d) \\ b + d & a + c \end{bmatrix} \\ &= \begin{bmatrix} a & 3b \\ b & a \end{bmatrix} + \begin{bmatrix} c & 3d \\ d & c \end{bmatrix} \\ &= \phi(a + b\sqrt{3}) + \phi(c + d\sqrt{3}). \\ \phi((a + b\sqrt{3})(c + d\sqrt{3})) &= \phi((ac + 3bd) + (ad + bc)\sqrt{3}) \\ &= \begin{bmatrix} ac + 3bd & 3(ad + bc) \\ ad + bc & ac + 3bd \end{bmatrix} \\ &= \begin{bmatrix} a & 3b \\ b & a \end{bmatrix} \begin{bmatrix} c & 3d \\ d & c \end{bmatrix}. \end{aligned}$$

We note $\ker \phi = \{0\}$, and that by definition, ϕ is onto. Therefore ϕ is a ring isomorphism.

16. We covered this one pretty carefully in class.

17. We also discussed this one quite carefully in class.

18. Suppose $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ are isomorphic, via the ring isomorphism $\varphi : \mathbb{Q}(\sqrt{3}) \rightarrow \mathbb{Q}(\sqrt{5})$. This is onto, so namely, by Proposition 7.1.9, $\varphi(1) = 1$, and thus $\varphi(3) = 3\varphi(1) = 3$. We notice that $\sqrt{3}$ solves the equation $x^2 - 3$ in $\mathbb{Q}(\sqrt{3})$, and thus, if $y = \varphi(\sqrt{3}) \in \mathbb{Q}(\sqrt{5})$, then

$$0 = \varphi(0) = \varphi((\sqrt{3})^2 - 3) = y^2 - 3.$$

But there is no element y which satisfies this property in $\mathbb{Q}(\sqrt{5})$, and thus, there is no such isomorphism.

19. We discussed this one in class.

20. (a) One checks $\varphi((a, b) + (c, d)) = \varphi(a + c, b + d) = a + c = \varphi(a, b) + \varphi(b, d)$. Also, $\varphi((a, b)(c, d)) = \varphi(ac, bd) = ac = \varphi(a, b)\varphi(c, d)$.

(b) We define $\varphi : R_1 \times R_2 \rightarrow R_2 \times R_1$ as $\varphi(a, b) = (b, a)$. One checks this is a ring homomorphism since $\varphi((a, b) + (c, d)) = \varphi(a + c, b + d) = (b + d, a + c) = (b, a) + (d, c) = \varphi(a, b) + \varphi(c, d)$, and $\varphi((a, b)(c, d)) = \varphi(ac, bd) = (bd, ac) = (b, a)(d, c) = \varphi(a, b)\varphi(c, d)$. We see $\ker \varphi = \{(0, 0)\}$ and this is clearly onto. Hence, this is a ring isomorphism.

23. We did this one in class. Consider the equation mod 13.