Homework # 1 Solutions, Section 6.1

A Handsome Energy Creature

April 20, 2009

Section 6.1: 1, 3–5, 8, 9, 11, 13, 17.

1. Notice that $S \subset \mathbb{R}$, and so we may use the subring test to check that S is a subring of $\mathbb{R}$, and thus a ring in its own right. For any $a, b, c, d \in \mathbb{Z}$,

$$(a + b\sqrt{3}) - (c + d\sqrt{3}) = (a - c) + (b - d)\sqrt{3} \in S,$$

and

$$(a + b\sqrt{3})(c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3} \in S.$$

So by the rubsing test, S is a subring of $\mathbb{R}$, and is thus a ring.

3. Notice that $S \subseteq M(2, \mathbb{R})$. We again use the subring test to show this is a ring. Let $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \in S$. Then

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} - \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} a - c & b - d \\ 0 & a - c \end{bmatrix} \in S,$$

and

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad + bc \\ 0 & ac \end{bmatrix} \in S.$$

So S is a subring of $M(2, \mathbb{R})$, and is thus a ring on its own.

4. Notice that $S \subseteq M(2,\mathbb{R})$. So we apply the same methods to show this is a subring, and thus a ring on its own. Let $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \in S$. Then

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} - \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -(b+d) & a+c \end{bmatrix} \in S,$$

and

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{bmatrix} \in S.$$

So S is a subring of $M(2,\mathbb{R})$ and is thus a ring in its own right.

5. This is not a ring, since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S$, but that

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S.$$

8. We show that $F(\mathbb{R})$ is a ring directly. First, we show that $(F(\mathbb{R}),+)$ is an abelian group. The operation $+$ is clearly closed. For $f, g \in F(\mathbb{R})$, and $x \in \mathbb{R}$, we have $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$. So the operation $+$ is commutative. Let 0 be the zero function, then $(0+f)(x) = (f+0)(x) = f(x)+0(x) = f(x)$. So 0 functions as the (two-sided) additive identity. The operation is associative since, for $f, g, h \in F(\mathbb{R})$, we have

$$\begin{aligned} (f+(g+h))(x) = f(x)+(g+h)(x) &= f(x)+(g(x)+h(x)) \\ &= (f(x)+g(x))+h(x) \\ &= (f+g)(x)+h(x) = ((f+g)+h)(x). \end{aligned}$$

Finally, we see that $(-f)(x) = -f(x)$ is the additive inverse of f, since $(f+(-f))(x) = f(x)-f(x) = 0$.

Next we verify the other ring axioms. We see that multiplication of functions is clearly closed, and that it is associative since

$$(f(gh))(x) = f(x)\cdot(gh)(x) = f(x)\cdot(g(x)\cdot h(x)) = (f(x)\cdot g(x))\cdot h(x) = (fg)(x)\cdot h(x) = ((fg)h)(x).$$

We see that multiplication is commutative, and so we only need to show that it distributes over addition on one side to complete the proof.

$$\begin{aligned} [f(g+h)](x) &= f(x)(g+h)(x) \\ &= f(x)[g(x)+h(x)] \\ &= f(x)g(x)+f(x)h(x) \\ &= (fg)(x)+(fh)(x) \\ &= (fg+fh)(x). \end{aligned}$$

9. (a) We showed in class that if R and S are any rings, then $R \times S$ is a ring. (actually, we carefully demonstrated this for $S = R$, but then I pointed out that the proof is exactly the same when S is another arbitrary ring.) So we prove that $R_1 \times \cdots \times R_n$ is a ring by induction. If $n = 1$, R_1 is a ring by hypothesis, and we even proved that for $n = 2$, $R_1 \times R_2$ is a ring. Now assume that, for some $n \geq 2$, that $R_1 \times \cdot \times R_n$ is a ring. Then using $R = R_1 \times \cdots \times R_n$, and $S = R_{n+1}$, we have $R \times S = R_1 \times \cdot \times R_n \times R_{n+1}$ is a ring.

(b) Suppose all of the R_i are commutative for $i = 1, 2, \ldots, n$. Then for $(r_1, \ldots, r_n), (s_1, \ldots, s_n) \in R_1 \times \cdots \times R_n$, we have

$$\begin{aligned}(r_1, \ldots, r_n) \cdot (s_1, \ldots, s_n) &= (r_1 s_1, \ldots, r_n s_n) \\ &= (s_1 r_1, \ldots, s_n r_n) \\ &= (s_1, \ldots, s_n) \cdot (r_1, \ldots, r_n).\end{aligned}$$

So under these hypotheses, $R_1 \times \cdots \times R_n$ is commutative.

We now show the contrapositive of the converse. Suppose for some i, that R_i is not commutative. Then there exists elements $r, s \in R_i$ so that $rs \neq sr$. Then it follows that $R_1 \times \cdots \times R_n$ is not commutative, since the following elements do not commute: Let r^i be the element of $R_1 \times \cdots \times R_n$ that consists of zeros in every coordinate except the i^{th} coordinate, and in that spot, we have r. Define s^i similarly. Then $r^i s^i \neq s^i r^i$, and S is not commutative.

11. We use the subring test to show that $S \cap T$ is a subring of R. Let $a, b \in S \cap T$. Then $a - b \in S$ and $a - b \in T$ since both elements are in both subrings S and T, and each subring is closed under addition. Similarly, $ab \in S$ and $ab \in T$ for the same reason. So $S \cap T$ is a subring of R.

13. We use the subring test. Let $a, b \in Z(R)$, and let $r \in R$ be arbitrary. Then $(a-b)r = ar - br = ra - rb = r(a-b)$, so $a - b \in Z(R)$. Similarly, $(ab)r = a(br) = a(rb) = (ar)b = (ra)b = r(ab)$. So $ab \in Z(R)$, and so by the subring test, $Z(R)$ is a subring of R.

17. We repeatedly use the distributive property to first expand $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= (a+b)a + (a+b)b \\ &= a^2 + ba + ab + b^2.\end{aligned}$$

Now if R is commutative, $ab = ba$, and so $(a+b)^2 = a^2 + 2ab + b^2$. Now suppose that $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$. Comparing our previous expansion to this new hypothesis, we have

$$\begin{aligned}(a+b)^2 &= a^2 + b^2 + 2ab \\ &= a^2 + b^2 + ab + ba.\end{aligned}$$

So we conclude that $2ab = ab + ab = ab + ba$. Subtracting, we have $ab = ba$.