HW # 8

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Hi everyone. Please work as efficiently as possible. I'm going to go get a fax from my future self.

1. Number 2: There are 7 nonidentity elements of order 2. Each of these elements, along with the identity of the group, form a subgroup of order 2, and each of these subgroups are distinct.

2. Number 5: We prove that $\mathbb{Z}\oplus\mathbb{Z}$ is not cyclic, which shows that (cyclic)$\oplus$(cyclic) need not be cyclic. Suppose that $\mathbb{Z}\oplus\mathbb{Z} =< (a, b) >$. Then $(0, b) \in< (a, b) >$, and so there exists a k so that $(0, b) = k(a, b) = (ka, kb)$, and so $0 = ka$ and $b = kb$. If $b \neq 0$, then $k = 1$. And if $k = 1$, then $a = 0$. So in either case, either a or b is zero. If $a = 0$, then the element $(1, 0) \notin< (0, b) >$, and a similar contradiction holds if $b = 0$. Thus $\mathbb{Z}\oplus\mathbb{Z}$ is not cyclic.

3. Number 12: Suppose that R is the cyclic subgroup generated by the rotations in D_n, and that F is any subgroup of order 2. Since R is cyclic, it is abelian, and since the order of F is prime, then F must be cyclic, and hence abelian. Then $R \oplus F$ is abelian, and thus it must not be isomorphic to the nonabelian group D_n.

4. Number 46: We map $\varphi(ax^2+bx+c)=(a,b,c)$. This is clearly well defined, and we check that it is an isomorphism. First, we observe

$$\begin{aligned}\varphi((ax^2+bx+c)+(dx^2+ex+f) &= \varphi((a+d)x^2+(b+e)x+(c+f)) \\ &= (a+d,b+e,c+f) \\ &= (a,b,c)+(d,e,f) \\ &= \varphi(ax^2+bx+c)+\varphi(dx^2+ex+f).\end{aligned}$$

So φ is a homomorphism. We can see that $(a,b,c)=\varphi(ax^2+bx+c)$ for any $(a,b,c)\in\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3$, and so φ is onto. Finally, if $\varphi(ax^2+bx+c)=(0,0,0)$, then it follows that $a=b=c=0$, and so as the kernal of φ is trivial, this homomorphism is one-to-one as well. Thus, it is an isomorphism.

The generalization comes from taking any abelian groups $G_0,\ldots,G_n$, and arranging them (symbolically) into a polynomial group as $G=G_nx^n+\cdots G_1x+G_0=\{g_nx^n+\cdots+g_1x+g_0|g_i\in G_i\}$, and that this group $G\cong G_0\oplus\cdots G_n$. Usually this is done in this setting where the G_i are all the same group, but sometimes not. ROCK ON!