Math 545 Homework Solutions # 6

Chopper

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Hi everyone! If anyone knows who I am, I'll be impressed. I'm a very small and meek reindeer that, for some reason, can change shapes into a huge and menacing reindeer. This seems to contradict the physical law of conservation of mass. Enjoy the solutions!

1. Number 3: We show that the mapping $\varphi(x) = \sqrt{x}$ is an automorphism of the group of positive real numbers under multiplication. This function is operation preserving, as $\varphi(xy) = \sqrt{xy} = \sqrt{x}\sqrt{y} = \varphi(x)\varphi(y)$. It's 1-1 since $\varphi(x) = \sqrt{x} = \sqrt{y} = \varphi(y) \Rightarrow x = y$. It's onto, for, if $z \in \mathbb{R}^+$, then $z^2 \in \mathbb{R}^+$, and $\varphi(z^2) = \sqrt{z^2} = |z| = z$. So φ is an automorphism.

2. Number 4: We observe that $U(8)$ is *not* cyclic, since it is a group of order 4, and every element has order 2. Next we observe that $U(10)$ *is* cyclic, as $U(10) =< 3 >$. An isomorphism between these groups would preserve the property of being cyclic, and since one is cyclic and one is not, these groups are not isomorphic.

3. Number 10: First we assume that G is abelian, and show that the map $\alpha(x) = x^{-1}$ is an automorphism. It is operation preserving since $\alpha(xy) = (xy)^{-1} = (yx)^{-1} = x^{-1}y^{-1} = \alpha(x)\alpha(y)$. It is 1-1 since $\alpha(x) = x^{-1} = y^{-1} = \alpha(y)$ if and only if $x = y$, and it is onto since$\alpha(y^{-1}) = (y^{-1})^{-1} = y$ for any $y \in G$. So α is an automorphism.

   Next we assume that α is an automorphism and show that G is abelian. Let $a, b \in G$. Then on the one hand, $\alpha(ab) = (ab)^{-1}$, but on the other hand, $\alpha(ab) = \alpha(a)\alpha(b) = a^{-1}b^{-1}$. So $(ab)^{-1} = a^{-1}b^{-1}$, and by inverting both sides, we see that $ab = (a^{-1}b^{-1})^{-1} = (b^{-1})^{-1}(a^{-1})^{-1} = ba$.

4. Number 35: First we point out that Corey seems to remember showing in class that $\varphi_a \varphi_b = \varphi_{ab}$. If this isn't the case, then please give Corey a head start to run before calling the people in the white coats. But using $a = b$ and repeatedly applying this property, we see that $[\varphi_a]^n = \varphi_{a^n}$ for $n \geq 2$, and using a^{-1} instead of a, and $b = a^{-1}$ shows that $[\varphi_a]^n = \varphi_{a^n}$ for $n \leq 0$. The case $n = 1$ is obvious, and using $b = e$ establishes the case for a^{-1} when $n = -1$. So $[\varphi_a]^n = \varphi_{a^n}$ holds for all n. The rest of the proof is straightforward.

   Let $|a| = k$. Then we see that $[\varphi_a]^k(x) = \varphi_{a^k}(x) = \varphi_e(x) = exe^{-1} = x$ for any $x \in G$. Thus $[\varphi_a]^k$ =the identity automorphism. So we conclude that whatever the order of this object is, $|\varphi_a| | k$.

   An example can be seen with $a = R_{90} \in D_4$. Notice that $|a| = 4$, and that $a^2 \in Z(D_4)$. So $[\varphi_a]^2(x) = a^2 x a^{-2} = x a^2 a^{-2} = x$ for all $x \in D_4$. So $|\varphi_a| \leq 2$. Since φ_a is not the identity automorphism since $a \notin Z(D_4)$, $|\varphi_a| > 1$. So $|\varphi_a| = 2$, and $1 < |\varphi_a| < |a|$.