

Math 545 Homework Chapter # 4 Solutions

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March 5, 2008

Hi everyone! I'm Corey's new niece, and I was born yesterday, March 4th, 2008! I sort of had a busy day yesterday at the hospital, and entertaining family. But today I managed to get away for a little bit and write your solutions today. Finally! Some alone time! And, of course, mathematics would be my choice of activity. ROCK ON!

1. Number 1: The generators for the cyclic groups $\mathbb{Z}_6, \mathbb{Z}_8$, and \mathbb{Z}_{20} are given by the set of integers (modulo 6, 8, and 20, respectively), which are relatively prime to 6 (for \mathbb{Z}_6), 8 (for \mathbb{Z}_8), and 20 (for \mathbb{Z}_{20}). So

$$\begin{aligned}\mathbb{Z}_6 &= \langle 1 \rangle = \langle 5 \rangle \\ \mathbb{Z}_8 &= \langle 1 \rangle = \langle 3 \rangle = \langle 5 \rangle = \langle 7 \rangle \\ \mathbb{Z}_{20} &= \langle 1 \rangle = \langle 3 \rangle = \langle 7 \rangle = \langle 9 \rangle = \langle 11 \rangle = \langle 13 \rangle = \langle 17 \rangle = \langle 19 \rangle .\end{aligned}$$

2. Number 21: (a) The possible orders are divisors of 12. (b) The possible orders are divisors of m . (c) Suppose G is cyclic, and that $a^8 \neq e \neq a^{12}$. Then we will show that the order of a is 24, and thus $G = \langle a \rangle$. The order of a must be a divisor of the order of G , so the possible orders of a are 1, 2, 3, 4, 6, 8, 12 or 24. Since $a^8 \neq e$, we know the order of a is not a divisor of 8, so the numbers 1, 2, 4, and 8 must not be the order of a . Since $a^{12} \neq e$, we may cross off the possible orders of 3 and 12 (the other divisors of 12 have already been eliminated). So the only possibility left is that $|a| = 24$.
3. Number 62: We show that H is a cyclic group, thus since it is a subset of $GL(2, \mathbb{R})$, by definition, it is a subgroup. We claim that

$$H = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

To see this, we use induction. Denote the matrix above as a_1 , and for shorthand, denote the matrix

$$a_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

. We wish to show that $a_1^n = a_n$ for all $n \in \mathbb{Z}$ by induction. Clearly $a_1^1 = a_1$, and that direct computation and the induction hypothesis shows that for $k > 1$, we have $a_1^{k+1} = a_1^k a_1 = a_k a_1 = a_{k+1}$. Clearly $a_1^0 = a_0 =$ the identity matrix, and one can check that $a_1^{-1} = a_{-1}$. Then we suppose that for some $k < -1$ we have $a_1^k = a_k$, and we again check manually that $a_1^{k-1} = a_1^k a_1^{-1} = a_k a_{-1} = a_{k-1}$. So by two applications of induction, we are done, and $H = \langle a_1 \rangle$ is a cyclic subgroup of $Gl(2, \mathbb{R})$.

4. G is a group under polynomial addition mod 3, since the operation is closed, associative, the polynomial 0 is the additive identity, and negating each coefficient will yield each element's inverse (so inverses exist). There are 3 possibilities for each of the 3 coefficients, so this group has $27 = 3^3$ elements. Now, if this group were cyclic, there would have to be an element of order 27. We show that each element has order 3, thus, this group must not be cyclic. Let $ax^2 + bx + c \in G$ is an arbitrary element (here, remember, a, b and c are elements of \mathbb{Z}^3), then $3(ax^2 + bx + c) = 3ax^2 + 3bx + 3c = 0$. Thus every element has order 3, and there are no elements of order 27, and it follows that this group is not cyclic.