Solutions to Selected Homework # 3 Problems

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Hi kids! Foamy here, delivering to you the abstract algebra you all crave so much from Chapter 3! I hope these solutions will help you in your quest to understand groups.

1. Number 2. By definition, we have

$$\begin{array}{lcl} \text{In } (\mathbb{Q},+): \langle \frac{1}{2} \rangle &=& \{0, \pm\frac{1}{2}, \pm 2\frac{1}{2}, \pm 3\frac{1}{2}, \ldots\} \\ \text{In } (\mathbb{Q}^*,\cdot): \langle \frac{1}{2} \rangle &=& \{1, \left(\frac{1}{2}\right)^{\pm 1}, \left(\frac{1}{2}\right)^{\pm 2}, \left(\frac{1}{2}\right)^{\pm 3}, \ldots\}. \end{array}$$

2. Number 10. Let $a, b \in G$, an abelian group, where $|a| = |b| = 2$. This implies that $a \neq e$, and $b \neq e$. Then we claim that the set $H = \{e, a, b, ab\}$ is a subgroup of G. We apply the finite subgroup test to this nonempty subset of G. Note that G is abelian, so we need only check the elements $a \cdot b = ab, a \cdot (ab) = a^2 b = b, b \cdot (ab) = bab = ab^2 = a$. The other products are redundancies created by the fact that G is abelian, or obvious products such as $e \cdot a = a$. So by the finite subgroup test, $H \leq G$.

   Now we count the elements of H. Since neither of a or b is the identity, the set H has 4 elements if $ab \neq b, ab \neq a$, and $ab \neq e$. If $ab = b$ or $ab = a$, then it follows that a or b is the identity, which is not the case. If $ab = e$ then $b = a^{-1}$, and since $a \cdot a = e, a^{-1} = a$. So $ab \neq e$ since if it did, $b = a^{-1} = a$, and we assumed that $a \neq b$.

3. Number 14. Suppose $H, K \leq G$. Then we prove that $H \cap K \leq G$. Notice first that $e \in H$ and $e \in K$, so that $H \cap K \neq \emptyset$ (Corey did not punish those who didn't specifically mention this, but I would like to point out that it is an important point to carefully check your hypotheses before applying any theorem or test. We wish to use the 1-step subgroup test, which requires the item being tested to be nonempty). Let $a, b \in H \cap K$. Then note that $b \in G$, so that there is a unique inverse $b^{-1} \in G$, and that this inverse must also be in both H and K. So it follows that a and b^{-1} are in both H and K, and so ab^{-1} is an element of both H and K. So by the 1-step subgroup test, $H \cap K \leq G$.

   Remark. Some of you said that $b \in H$ implies $b^{-1} \in H$, and that $b \in K$ implies $b^{-1} \in K$. This is true. But then some went on to say that this implies b^{-1} is in H and K. Now you have invoked the closure of (sub)groups of taking identities, but the inverse you would get wouldn't necessarily be the same inverse in both groups. Take, for instance, the identity matrix with 1's down the diagonal, and 0's elsewhere. The multiplicative inverse of this is itself, while the additive inverse of this is the diagonal matrix with -1's down the diagonal. So this is a situation where the inverse of the same element is two different element. And this is an artifact of that given element being an element of two different groups that happen to share some elements. The fact that b^{-1} is the same element in each of the subgroups H and K is that they are both subgroups of the same group G, and that b^{-1} is a unique element in G. Thus it must be the same inverse in H as it is in K. This is a minor point, but worth remarking on.

4. Number 44. Let $A, B \in H$. Then $det(A) = 2^a$ and $det(B) = 2^b$. Then $det(AB^{-1}) = det(A)/det(B) = 2^a/2^b = 2^{a-b}$, so that $AB^{-1} \in H$. Thus, by the 1-step subgroup test, $H \leq Gl(2, \mathbb{R})$.

ROCK ON!!!!