Solutions to HW # 2

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Hello, fellow group theorists! Here are some thoughts on the homework # 2 assignment, from Chapter 2. Rock on!

1. Number 1: The set of odd integers under addition, were it a group, would be a subgroup of $\mathbb{Z}$, as it is a subset with the same operation of addition. Thus the identity element 0 would have to be the identity of this subgroup, but 0 is not odd. Also, closure fails: the sum of two odd integers is even.

2. Number 5: We use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

   where we interpret the fraction $\frac{1}{ad-bc}$ as multiplication by the inverse of $ad-bc$ in the coefficients given. In this case, $\mathbb{Z}_{11}$. So we compute the determinant

$$det\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix} = -8 = 3 \; mod \; 11.$$

   And $3^{-1} = 4$ in $\mathbb{Z}_{11}$. So the inverse would be

$$\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}^{-1} = 4\begin{bmatrix} 5 & -6 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 10 & 8 \end{bmatrix}$$

3. Number 15: Let $n \geq 1$. Then clearly $(ab)^1 = a^1b^1$. Suppose that for some $k \geq 1$ we have $(ab)^k = a^kb^k$. Then $(ab)^{k+1} = (ab)(ab)^k = (ab)a^kb^k = aa^kbb^k = a^{k+1}b^{k+1}$,

where we have used equality, the induction hypothesis, the property of being abelian, and again equality to justify these steps (respectively). So the property holds for all positive integers. It holds for $n = 0$ since $(ab)^0 = e = a^0 b^0$. Now suppose that $n \leq -1$, and set $m = -n \geq 1$. Then we notice that $(ab)^n = (ab)^{-m} = [(ab)^m]^{-1} = [a^m b^m]^{-1} = b^{-m} a^{-m} = a^n b^n$. We justify these steps by $n = -m$, equality of elements, using our fact that we just showed about positive integers, the socks and shoes property, and lastly equality.

4. Number 33: Suppose $a^2 = b^2 = (ab)^2 = e$. Then $ee = a^2 b^2 = (ab)^2 = e$, and so $aabb = abab$, and by canceling a on the left and b on the right, we have $ab = ba$.