Name: ______________________________

STOLEN Math 531 Final Exam of Fun Springtime!!!

Directions: Please take this exam without cheating. It's worth 200 points. Please also read the directions to each question carefully–questions that you do not follow the directions on may result in a score of zero points for that question. For this exam, always assume that V is a vector space, and that the dimension of V is finite. Also, T is a linear operator, and E_λ and K_λ are the eigenspace and generalized eigenspace of T corresponding to the eigenvalue λ. All other notation will be the notation we've developed throughout the course. Anything else which is unclear will be well defined for you. You may use a writing utensil, a calculator, and your brain. Oh, and ROCK ON!!!

1. (25 points) Please do exactly two of the following.
   (a) Let T be a linear operator. Please carefully define what it means for λ to be .... Please also carefully define what it means for .... And finally, please show that if $\{v_1, \ldots, v_k\}$ are eigenvectors of T corresponding to different eigenvalues (i.e., $v_i \in E_{\lambda_i}$, and $\lambda_i \neq \lambda_j$), then ....
   (b) Please carefully define the spaces .... and K_λ. Please also show that if $0 \neq v_\lambda \in K_\lambda$ and $0 \neq v_\eta \in K_\eta$ where $\lambda \neq \eta$, then the set $\{v_\lambda, v_\eta\}$ is linearly independent.
   (c) Please carefully define..................... Please also show that for any $x, y \in V$ that $T^*(x+y) = T^*(x) + T^*(y)$.
   (d) Please carefully define what it means for T to be a normal operator. Please also ....
   (e) Please carefully define the Jordan block $J(\lambda, k)$. Please also carefully define ....
2. (35 points) Please do exactly one of the following.
   (a) For this problem, define A as follows:

   ....

   Please do all of the following.
      i. Please compute all of the .... Please also ....
      ii. For each eigenvalue λ, please compute the dimension ....
      iii. ....? If so, please.................... If not, please describe why.
   (b) Suppose T and U are linear operators, $Spec(T) = Spec(U)$, and that $|Spec(T)| = |Spec(U)| = \dim(V)$. Under these assumptions, please do both of the following.
      i. True or False: "...................." If so, please prove that they are. If not, the please provide a counterexample.
      ii. Suppose that $Rank(T - \lambda I) \neq Rank(U - \lambda I)$ for some eigenvalue λ. ....?
   (c) Suppose $\lambda, \eta \in Spec(T)$, with .... Let $\beta = \{v_1, v_2\}$, and let $\gamma = \{w_1, w_2\}$. If β and γ are linearly independent, ...
3. (35 points) Please do exactly one of the following.
   (a) Show T is .... if and only if $V = W_1 \oplus \cdots \oplus W_n$, where....................
   (b) Let λ be an eigenvalue of T, and let m be the multiplicity of λ. Please .... Please give an example where the inequality is strict.
   (c) Let S and T be subspaces of V, let β_S be a basis for S, and let β_T be a basis for T. Please show that ....
   (d) Let W be a $T-$invariant subspace of V, and let T_W be the operator T restricted to W. ....
4. (35 points) Please do exactly one of the following.

(a) Suppose that $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are both inner products for the same vector space V. .... (that is, they are positive real numbers), then ....

(b) Please do all of the following. Assume that $\langle \cdot, \cdot \rangle$ is an inner product on V.

   i. Show that if .... for all $y \in V$, then ....

   ii. Suppose that the set $\{v_1, \ldots, v_k\}$ is an orthogonal set of vectors. Show that it is also ....

   iii. Suppose T is a linear operator on V, and that the set $\beta = \{v_1, \ldots, v_n\}$ is an orthonormal basis for V. If A_{ij} is the (i, j) entry of $[T]_\beta$, then prove ....

(c) Suppose β is an orthonormal basis for V. ....

(d) Let g be any polynomial, and suppose that W is $T-$invariant. ....

5. (35 points) Please do exactly one of the following.

(a) Please do all of the following. Assume V is an inner product space.

   i. Suppose W is a subspace of V, and that β_W is .... Show that $x \in W^\perp$ if and only if ....

   ii. .... Please prove that $||Tx|| = ||T^*x||$ for all $x \in V$.

   iii. Please show that if T is normal, then ....

(b) .... $K_\eta \subseteq K_\lambda^\perp$.

(c) Suppose T is an orthogonal operator on a real vector space. Please do both of the following.

   i. ....

   ii. ....

6. (35 points) Please do exactly one of the following.

(a) Suppose T and U are self-adjoint operators on a real vector space. .... there exists an orthonormal basis of eigenvectors of both T and U.

(b) ....

(c) Suppose $T_1^2 = T_2^2 = 0$, where T_1 and T_2 are linear operators on V. Show that T_1 is Jordan equivalent to T_2 ....

(d) Suppose that V is an inner product space of dimension 3, and that there exists a basis β so that $[T]_\beta = J(\lambda, 3)$. Show that if ...., then β is *not* an orthonormal basis. (Hint: Show that the.................... fails by assuming that .... and finding a vector x so that ....)

7. For 2 free points, tell me something funny!