Exam Solutions

The Northern Lights

February 9, 2010

Hi everyone! The Northern Lights here, coming to you from northern Finland, bringing you the solutions to your most recent exam. Enjoy!

1. (a) and (b) can be found as definitions in the book. So can the first part of (c). For the second part of (c): let β_W be a basis for W, and extend this basis to be a basis β for V. Since W is T-invariant, we have $Tw \in W$ for all $w \in W$. Thus, the matrix representation for T with respect to the basis β is

$$[T]_\beta = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

One sees that the submatrix $A = [T_W]_{\beta_W}$. Thus, for the symbol I below as the identity matrix of appropriate size,

$$Char(T) = \det(T - tI) = \det \begin{bmatrix} A - tI & B \\ 0 & C - tI \end{bmatrix} = \det(A - tI) \cdot \det(C - tI) = Char(T_w) \cdot g(t),$$

for some polynomial $g(t)$. Thus, $Char(T_W) | Char(T)$.

As for part (d), the first part may be found as a definition in the book, and for the second part, if $v \in E_\lambda \cap E_\eta$, then $Tv = \lambda v = \eta v$, and so $(\lambda - \eta)v = 0$. But $\lambda \neq \eta$, so $v = 0$.

2. (a) (i) We compute the characteristic polynomial:

$$\det(A - tI) = \det \begin{bmatrix} -t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -t \end{bmatrix} = -t^3 + 3t + 2 = (2-t)(t+1)^2.$$

So the eigenvalues are $t = 2$ with multiplicity 1, and $t = -1$ with multiplicity 2.

(ii) We start with the eigenvalue $t = -1$, and consider the matrix

$$A - (-1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

This matrix has rank 1, so that the dimension of the kernal of $A - (-1)I$ (or, its nullity) is 2, which is equal to the multiplicity of the eigenvalue -1. Since 2 is an eigenvalue, the dimension of E_2 is at least 1. It is also at *most* 1, since, if it were any higher, for dimensional reasons, $E_{-1} \cap E_2 \neq \{0\}$, a contradiction to a question on the first page of this exam.

(iii) It follows that A is diagonalizable. One does elementary row reductions to find $\beta_{-1} = \{(-1, 0, 1), (-1, 1, 0)\}$ as a basis for E_{-1}, and one sees that $\beta_2 = \{(1, 1, 1)\}$ is a basis for E_2. Thus, A, when written with respect to that basis, must be the diagonal matrix $diag(-1, -1, 2)$.

(b) We proceed by induction. Since the vectors v_i are eigenvectors, they are necessarily nonzero. Thus, for $k = 1$, the set $\{v_1\}$ is linearly independent. Suppose any subset of $k - 1$ eigenvectors corresponding to different eigenvalues were linearly independent, and consider $\sum_{i=1}^k a_i v_i = 0$. We apply the operator $T - \lambda_k I$ to get

$$(T - \lambda_k I) \sum_{i=1}^{k} a_i v_i = 0 = \sum_{i=1}^{k} a_i (T - \lambda_k I) v_i = \sum_{i=1}^{k} a_i (\lambda_i - \lambda_k) v_i = \sum_{i=1}^{k-1} a_i (\lambda_i - \lambda_k) v_i.$$

This is a null linear combination of the (assumed) linearly independent vectors $v_1, \ldots, v_{k-1}$. So, the coefficients $a_i(\lambda_i - \lambda_k) = 0$. But the eigenvalues are all distinct, so one concludes that $a_1 = \cdots = a_{k-1} = 0$. But now we use that information back into the sum $\sum_{i=1}^k a_i v_i = 0$ to conclude $a_k v_k = 0$. Since $v_k \neq 0$, it follows that the last coefficient $a_k = 0$ as well.

(c) Since $\mathcal{J}(\lambda, k)$ is upper triangular, its eigenvalues are its diagonal entries. There is only one diagonal entry λ, repeated k times. Thus, the characteristic polynomial of $\mathcal{J}(\lambda, k)$ splits. But, to be diagonalizable, one could then check if the dimension of the eigenspace E_λ is equal to the multiplicity of λ, which is k. There are now two ways to proceed. One is a little shorter than the other, but either would work.

One could note that if $\mathcal{J}(\lambda, k)$ were diagonalizable, then it would be similar to the scalar matrix λI, and we proved in our homework that it would follow that $\mathcal{J}(\lambda, k) = \lambda I$, which it most surely does not. It follows that $\mathcal{J}(\lambda, k)$ is not diagonalizable.

Otherwise, one could analyze $\dim(\ker(\mathcal{J}(\lambda, k) - \lambda I)) = \dim E_\lambda$. The resulting matrix is already in reduced row echelon form, and clearly has rank $k-1$, which means $\dim E_\lambda = 1$ by the rank-nullity theorem. One needn't be so careful about it…the question is whether or not $\dim E_\lambda = k$, which would mean $Rank(\mathcal{J}(\lambda, k) - \lambda I) = 0$, which would mean $\mathcal{J}(\lambda, k) - \lambda I = 0$, or that $\mathcal{J}(\lambda, k) = \lambda I$, another proof of the fact used in the most previous paragraph. So, again, since this is not the case, $\mathcal{J}(\lambda, k)$ is not diagonalizable.

3. (a) For the first part, see the Corollary on page 261. The converse is not true. The scalar matrix λI is diagonalizable, but its eigenvalues are all the same.

   (b) (i) Suppose λ is an eigenvalue of A. Then there exists a (nonzero) eigenvector v with $Av = \lambda v$. Then repeatedly applying A gives us $0 = A^k v = \lambda^k v$, so that λ must be 0. For part (ii), suppose $A^k = 0$, and that A is diagonalizable. Then there exists a matrix Q with $diag(\lambda_1, \ldots, \lambda_n) = Q^{-1}AQ$. Thus, raising everything to the power k, we have
$$diag(\lambda_1, \ldots, \lambda_k)^k = diag(\lambda_1^k, \ldots, \lambda_n^k) = Q^{-1}A^kQ = 0.$$

   So it must be the case that $\lambda_i^k = 0$ for all i, and so $\lambda_i = 0$. So the diagonal matrix that A is similar to is the zero matrix. But by rearranging the Q's in the formula $0 = D = Q^{-1}AQ$, we have $Q0Q^{-1} = 0 = A$, a contradiction to the hypothesis $A \neq 0$.
   (c) This is just another way to state one of our homework questions. Suppose T is diagonalizable, and that the basis γ gives us $[T]_\gamma = D$, where D is the diagonal matrix with T's eigenvalues down the diagonal. λ is assumed to be the only eigenvalue, so $D = \lambda I$. Let β be *any* basis, and suppose Q is the transition matrix from γ to β. Then $[T]_\beta = Q^{-1}[T]_\gamma Q = Q^{-1}\lambda I Q = \lambda I$.

4. (a) Let the multiplicity of λ be m. Consider E_λ as the corresponding eigenspace, and that $\dim E_\lambda = d$. We must show that $1 \leq d \leq m$. Clearly, since λ is already an eigenvalue, $E_\lambda \neq \{0\}$, so that $d \geq 1$. We must show that $d \leq m$.

   The eigenspaces are T invariant, since, for $v \in E_\lambda$, $Tv = \lambda v \in E_\lambda$ because $T(\lambda v) = \lambda Tv = \lambda(\lambda v)$, so that if $w = \lambda v$, then $Tw = \lambda w$. According to the first page of the exam, $Char(T_{E_\lambda}) | Char(T)$. But because $Tv = \lambda v$ for every $v \in E_\lambda$, by part (a) of number 3,
$$T_{E_\lambda} = \lambda I,$$

   where I is the $d \times d$ identity matrix. So $Char(T_{E_\lambda}) = (\lambda - t)^d$. So $(\lambda - t)^d | Char(T)$. But m is by definition the *largest* integer k for which $(\lambda - t)^k | Char(T)$. So $d \leq m$.

(b) (i) The characteristic polynomial

$$Char(A) = \det(A - tI) = t^2 - 5t + 2.$$

(ii) Let $v = e_1$. Then $Ae_1 = e_1 + e_2 \notin spane_1$. So $\dim(W) \geq 2$. But $W \subseteq V$, so $\dim(W) \leq 2$. So a basis for W is $\{e_1, Ae_1\}$.

(iii) One can see that $A^2e_1 \in span\{e_1, Ae_1\} = V$, and that

$$2e_1 - 5Ae_1 + A^2e_1 = 0.$$

(iv) I am smiling at my screen right now.

(c) The matrix is upper triangular, so its eigenvalues are a and b. We diagonalize the matrix with the eigenvector $(1, 0)$ for the eigenvalue a, and with $(1, 1)$ for b. So, the transition matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ so } Q^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

It follows that $diag(a, b) = Q^{-1}AQ$, and that $Qdiag(a, b)Q^{-1} = A$. Thus

$$\begin{aligned} A^k &= Q \; diag(a^k, b^k) \; Q^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a^k & b^k - a^k \\ 0 & b^k \end{bmatrix}. \end{aligned}$$