

## HW 4 solutions

March Madness!!!!

March 16, 2010

*Here are solutions to the fourth homework assignment. Corey is hard at work studying NCAA basketball and filling out his bracket(s), so that means that I'm typing the solutions. Enjoy!*

6.4.4 The operator  $TU$  is self-adjoint if and only if  $TU = (TU)^* = U^*T^* = UT$ .

6.4.6 (a) Notice  $T_1^* = \frac{1}{2}(T + T^*)^* = \frac{1}{2}(T^* + T) = T_1$ , so that  $T_1$  is self-adjoint. For  $T_2$ , we have

$$\begin{aligned} T_2^* &= \overline{\left(\frac{1}{2i}\right)}(T - T^*)^* \\ &= -\frac{1}{2i}(T^* - T) \\ &= T_2. \end{aligned}$$

That  $T = T_1 + iT_2$  is an easy plug-and-chug.

(b) Suppose  $T = U_1 + iU_2$  and that  $U_1$  and  $U_2$  are self-adjoint. Then  $T^* = (U_1 + iU_2)^* = U_1^* + (iU_2)^* = U_1 - iU_2$ . Suppose also that  $T = T_1 + iT_2$  where the  $T_i$  are given in the book. Then

$$T_1 = \frac{1}{2}(T + T^*) = \frac{1}{2}(U_1 + iU_2 + U_1 - iU_2) = U_1.$$

Now  $T = T_1 + iT_2 = T_1 + iU_2$ , and it follows that  $T_2 = U_2$ .

(c) Suppose  $T$  is normal. Then we see that  $T_1T_2 = T_2T_1$  by the following computation:

$$\begin{aligned} T_1T_2 &= \frac{1}{2}(T + T^*)\frac{1}{2i}(T - T^*) \\ &= \frac{1}{4i}(T^2 - TT^* + T^*T - (T^*)^2) \\ &= \frac{1}{4i}(T^2 - (T^*)^2). \\ T_2T_1 &= \frac{1}{2i}(T - T^*)\frac{1}{2}(T + T^*) \\ &= \frac{1}{4i}(T^2 + TT^* - T^*T - (T^*)^2) \\ &= \frac{1}{4i}(T^2 - (T^*)^2). \end{aligned}$$

Conversely, suppose  $T_1T_2 = T_2T_1$ . Then we see  $TT^* = T^*T$  is normal by the following computation:

$$\begin{aligned} TT^* &= (T_1 + iT_2)(T_1 - iT_2) \\ &= T_1^2 - iT_1T_2 + iT_2T_1 + T_2^2 \\ &= T_1^2 + T_2^2. \\ T^*T &= (T_1 - iT_2)(T_1 + iT_2) \\ &= T_1^2 + iT_1T_2 - iT_2T_1 + T_2^2 \\ &= T_1^2 + T_2^2. \end{aligned}$$

6.4.7 (a) Since  $T$  is self-adjoint, it is diagonalizable (Theorem 6.16 or 6.17). Thus, its characteristic polynomial must split. Since  $W$  is a  $T$ -invariant subspace, and  $\text{Char}(T_W)$  divides  $\text{Char}(T)$ , we conclude that  $\text{Char}(T_W)$  splits as well. Therefore we may use Schur's Lemma to find an orthonormal basis  $\beta_W$  for  $W$  such that  $[T_W]_{\beta_W}$  is upper triangular. Extending  $\beta_W$  to be an orthonormal basis for  $W^\perp$ , one finds an ordered orthonormal basis  $\beta$  which is the union of  $\beta_W$  with an orthonormal basis for  $W^\perp$  (where, say, the elements of  $\beta_W$  come first). Let  $d = \dim W$ . So we observe the following about  $[T]_\beta$ . We know that the first  $d$  columns of  $[T]_\beta$  is an upper triangular  $d \times d$  matrix, with an  $(\dim V - d) \times d$  block of 0's beneath it. Since  $T$  is self-adjoint and  $\beta$  is orthonormal,  $[T]_\beta^* = [T]_\beta$ . Thus, the zeros beneath the diagonal in  $[T]_\beta$  must also be zeros to the *right* of the diagonal. That is to say, the upper  $d \times d$  submatrix of  $[T]_\beta$  is diagonal, and those diagonal entries are some of the eigenvalues of  $T$ . Since  $T$  is self-adjoint, those eigenvalues must be real. But, that  $d \times d$  submatrix is exactly  $[T_W]_{\beta_W}$ , and so we see that  $[T]_{\beta_W}^* = [T_W]_{\beta_W}$  with respect to the orthonormal basis  $\beta_W$ . Thus,  $T_W$  is self-adjoint.

(b) Let  $w \in W^\perp$ . Then we note that  $T^*w \in W^\perp$  since, for any  $z \in W$ , we have  $\langle z, T^*w \rangle = \langle Tz, w \rangle$ , and since  $W$  is  $T$ -invariant and  $z \in W$ , then  $Tz \in W$ . So  $\langle Tz, w \rangle = 0$ , and  $T^*w \in W^\perp$ .

(c) Let  $x, z \in W$ . We compute  $\langle Tx, z \rangle$  in two ways. First, since  $x \in W$ , and  $W$  is  $T$ -invariant, then  $Tx = T_Wx$ . So that

$$\langle Tx, z \rangle = \langle (T_W)x, z \rangle = \langle x, (T_W)^*z \rangle.$$

On the other hand,  $\langle Tx, z \rangle = \langle x, T^*z \rangle$ . Since  $W$  is also  $T^*$  invariant,  $T^*z = (T^*)_Wz$ , so that

$$\langle Tx, z \rangle = \langle x, T^*z \rangle = \langle x, (T^*)_Wz \rangle.$$

Equating these two, and using Theorem 6.1 (e) from Section 6.1, we have  $(T_W)^*z = (T^*)_Wz$  for all  $z \in W$ , so  $(T_W)^* = (T^*)_W$ .

(d) Assume  $TT^* = T^*T$ . We use the last result to compute

$$(T_W)(T_W)^* = (T_W)(T^*)_W = (TT^*)_W = (T^*T)_W = (T^*)_W(T_W) = (T_W)^*(T_W).$$

6.4.12 Let  $\dim(V) = n$ . Suppose that  $\text{Char}(T)$  splits. Then Schur's Theorem applies, and there exists an orthonormal basis  $\beta$  for  $V$  so that  $A = [T]_\beta$  is upper triangular. But  $T$  is also normal, and our field is the real numbers, so  $A^* = A^T$ , so  $AA^T = A^T A$ . Consider the  $(i, i)$  entry of the product  $AA^T$ . If  $a_{ij}$  is the  $(i, j)$  entry of  $A$ , then the  $(i, i)$  entry of  $AA^*$  is  $\sum_k a_{ik}^2$ . Since  $A$  is upper triangular,  $a_{ik} = 0$  if  $k < i$ . This must be equal to the  $(i, i)$  entry of  $A^*A$ , which is  $\sum_k a_{ki}^2$ . But  $A$  is upper triangular, and thus  $a_{ki} = 0$  for all  $k > i$ . So

$$\sum_{k \geq i} a_{ik}^2 = \sum_{k \leq i} a_{ki}^2.$$

When  $i = 1$ , then this amounts to  $\sum_{k=1}^n a_{1k}^2 = a_{11}^2$ . So,  $\sum_{k=2}^n a_{1k}^2 = 0$ . Since our field is the real numbers, this must mean that  $a_{1k} = 0$  for all  $k > 1$ . If  $i = 2$ , then since  $a_{1k} = 0$  for  $k > 1$ , the above equation results in

$$\sum_{k \geq 2} a_{2k}^2 = \sum_{k \leq 2} a_{k2}^2 = a_{22}^2.$$

Now, as before, one has  $\sum_{k=3}^n a_{2k}^2 = 0$  and so  $a_{k2} = 0$  for  $k > 2$ . Continuing in this process one finds that  $a_{ki} = 0$  for  $k > i$ . Since  $A$  is assumed to be upper triangular, we have  $a_{ki} = 0$  for  $k < i$ . So  $A$  must be diagonal. Thus, our basis  $\beta$  is an orthonormal basis of eigenvectors. By Theorem 6.17, this means that  $T$  is self-adjoint.

*Remark:* This problem really doesn't involve that much work, provided you believe (and can prove) a helpful Lemma that I've done in the context of the proof above. That is: If  $A$  is normal and triangular, then  $A$  is diagonal. From there, it is just an application of Schur's Theorem.

6.4.14 We will induct on the number of eigenvalues of  $T$ . If there is only one eigenvalue, since  $T$  is self-adjoint, it is diagonalizable, and  $[T]_\beta = \lambda I$ , for *any* basis  $\beta$ . Since  $U$  is self-adjoint, there exists an orthonormal basis of eigenvectors that will simultaneously diagonalize  $T$  and  $U$ .

Now suppose  $\text{Spec}(T) = \{\lambda_1, \dots, \lambda_k\}$ . We have already showed that  $E_\lambda$  is  $U$ -invariant for any  $\lambda \in \text{Spec}(T)$ . By problem number 7 of this section,  $U_{E_\lambda}$  is self-adjoint because  $U$  is. So there exists an orthonormal basis  $\beta_\lambda$  of eigenvectors for both  $T_{E_\lambda}$  and  $U_{E_\lambda}$ .

Set  $W = E_\lambda$ . Notice that  $W^\perp$  is  $T^*$ -invariant and  $U^*$ -invariant by Exercise 7. Since  $T$  and  $U$  are self-adjoint, then  $W^\perp$  is  $T$ - and  $U$ -invariant.

Consider the operator  $T_W$ . It is self-adjoint by Exercise 7. Since  $W = E_\lambda \perp W^\perp$ , it follows that  $\lambda \notin \text{Spec}(T_W)$ , and that since  $\text{Char}(T_W) | \text{Char}(T)$ , that the operator  $T_W$  has at most  $k - 1$  eigenvalues. Thus using the induction hypothesis, we may simultaneously diagonalize the operators  $T_{W^\perp}$  and  $U_{W^\perp}$  with an orthonormal basis  $\beta_{W^\perp}$  of (simultaneous) eigenvectors. The ordered basis  $\beta_\lambda \cup \beta_{W^\perp}$  is an orthonormal

basis  $(\beta_\lambda \perp \beta_{W^\perp})$  of simultaneous eigenvectors since  $V = W \oplus W^\perp$ . (It seems that the reference to Theorem 6.6 on page 350 simply references that  $V = W \oplus W^\perp$ , so that the union of the bases  $\beta_\lambda$  and  $\beta_{W^\perp}$  form a basis for  $V$ ).

6.6.6 Suppose that  $V = W_1 \oplus W_2$ , and that  $T$  is projection onto  $W_1$  along  $W_2$ . That is, whenever  $v = w_1 + w_2$  and  $w_i \in W_i$ , then  $Tv = w_1$ . Thus,  $W_1 = \text{Im}(T)$ , and  $W_2 = \ker(T)$ . We must show that  $W_2 \perp W_1$  to show, by definition, that  $T$  is an orthogonal projection. This will follow from the following fact.

We prove that if  $T$  is normal, then  $\|Tx\| = \|T^*x\|$ . This follows from the calculation

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle = \langle x, TT^*x \rangle = \langle T^*x, T^*x \rangle = \|T^*x\|^2.$$

That is to say, that if  $x \in \ker(T)$ , then  $x \in \ker(T^*)$ . So suppose  $w_i \in W_i$ . We must show that  $w_1 \perp w_2$ . Note that  $W_1 = \text{Im}(T)$ , in fact,  $W_1$  is  $T$ -invariant and  $T|_{W_1} = I$ , the identity map. Sure enough, if  $w_1 \in W_1$ , then  $Tw_1 = w_1$ . So, since  $w_2 \in \ker(T)$  we have  $w_2 \in \ker(T^*)$ , so

$$\langle w_1, w_2 \rangle = \langle Tw_1, w_2 \rangle = \langle w_1, T^*w_2 \rangle = \langle w_1, 0 \rangle = 0.$$

6.6.8 Suppose that  $U$  commutes with  $T$ . By the spectral theorem,  $T^* = g(T)$  for some polynomial  $g$ . Then since  $U$  commutes with  $T$ ,  $U$  commutes with  $g(T)$ , so that  $T^*U = g(T)U = Ug(T) = UT^*$ .