Math 355 Quiz # 1 Solutions of awesomeness.

A family of ducks!

May 1, 2008

Hi kids! Here are the solutions to the Quiz # 1 that you just took. I hope you like them! Generally, people did fine, but Corey would like to stress that a lot of you wrote in incomplete sentences, and forgot many other essential pieces of punctuation that make the English language understandable. And, if this is the course where you are to express your ideas clearly, then you must *get in the habit of writing clearly and professionally. Corey is a jerk about this. Ask any of his grad students, they'll say the same thing. He really didn't hit anyone hard on the quiz for not adhering to proper English, grammar, and punctuation unless it really blurred the meaning of what was on the page. But to be sure, he will in the future. So be sure to keep that in mind as you study for the upcoming exam. Otherwise,* ROCK ON!

1. Suppose $\sqrt{2}$ is rational. Then there exists integers $a, b \in \mathbb{Z}$ with $b \neq 0$ so that $\sqrt{2} = \frac{a}{b}$. Furthermore, a and b may be chosen so that they share no common factors. Then it follows that $2 = \frac{a^2}{b^2}$ and that $2b^2 = a^2$. Thus, a^2 is even. We claim that a must be even, for if a is odd, then a^2 would be odd, which contradicts a^2 being even. So there exists an integer $k \in \mathbb{Z}$ so that $a = 2k$. Then $a^2 = 4k^2$, and $2b^2 = 4k^2$, and $b^2 = 2k^2$, and so b^2 is even. By the same reasoning above, it follows that b is even. But we have concluded that both a and b must be even, a contradiction to a and b sharing no common factors. So no such a and b exist, and $\sqrt{2}$ is irrational.

2. We establish the assertion by double-inclusion. First assume that $x \in (A \cup B)^c$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$. So $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$. To prove the reverse inclusion, suppose $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$, so $x \notin A \cup B$, and so $x \in (A \cup B)^c$.

3. Let $y \in f(A \cap B)$. Then there exists an $x \in A \cap B$ so that $f(x) = y$. Then $x \in A$ and $x \in B$, and hence $f(x) = y$ is an element of $f(A)$ and $f(B)$. So $y \in f(A) \cap f(B)$.

4. (a) We prove this by induction. Let $n = 1$. By observation, $\sum_{k=1}^{1} k = 1$, as does $\frac{1(1+1)}{2} = 1$. Now, we assume the result holds for some natural number n, and we prove that the result also holds for $n + 1$. So we have

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\frac{n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

(b) We prove this by induction. Notice that $2^4 = 16$ and that $4! = 24$, so that $2^4 \leq 4!$, and the base case is established. Now suppose that the result holds for some natural number $k \geq 4$, and that we wish to show that the same result holds for $k+1$. That is, for some integer $k \geq 4$, we will assume that $2^k \leq k!$. Then we remark that $2 \leq k + 1$ before proving

$$2^{k+1} = 2 \cdot 2^k \leq 2 \cdot k! \leq (k+1)k! = (k+1)!.$$