Midterm Solutions!

Space Pants!

May 14, 2008

Hi everyone! I'm a puppet screaming "Space Pants!" at the top of my lungs. I hope you enjoy these solutions to the midterm!

1. The definitions may be found in the book, or certainly in the notes. For examples in each of these, we note for part (a) that if $A = \mathbb{Z}$, then $\sup(A) = \infty$. For part (b), we may note that $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{N} \neq \mathbb{Z}$, but that $\mathbb{N} \sim \mathbb{Z}$.

2. (a) We prove this by double-inclusion. Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$. So $x \notin A$ or $x \notin B$, and thus, $x \in A^c$ or $x \in B^c$. So $x \in A^c \cup B^c$, and thus $(A \cap B)^c \subseteq A^c \cup B^c$. Now suppose $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, so that $x \notin A$ or $x \notin B$. So $x \notin A \cap B$, or $x \in (A \cap B)^c$, and it follows that $A^c \cup B^c \subseteq (A \cap B)^c$.

   (b) Let $y \in f(A \cap B)$. Then there exists an $x \in A \cap B$ so that $f(x) = y$. But $x \in A \cap B$ means $x \in A$ and $x \in B$. Thus $f(x) = y \in f(A)$, and $f(x) = y \in f(B)$, so that $y \in f(A \cap B)$.

3. (a) We prove this fact by induction. If $n = 1$, then $1 = 1^2$, and the base case is established. Now suppose that for some $n \geq 1$, we have

   $$1 + 3 + \cdots + (2n - 1) = n^2.$$

   We wish to prove that

   $$1 + 3 + \cdots + (2n - 1) + (2n + 1) = (n + 1)^2.$$

But by adding $2n+1$ to the first equation gives us:

$$\begin{aligned} 1+3+\cdots+(2n-1)+(2n+1) &= n^2+2n+1 \\ &= (n+1)^2. \end{aligned}$$

(b) We prove this fact by induction. If $n=1$, then $7^1-3^1=4$, and 4 divides 4. Now we suppose that $4|(7^n-3^n)$ for some $n \geq 1$. We wish to show that $4|(7^{n+1}-3^{n+1})$. But

$$\begin{aligned} 7^{n+1}-3^{n+1} &= 7\cdot 7^n - 3\cdot 3^n \\ &= 7\cdot 7^n-(7-4)3^n \\ &= 7(7^n-3^n)-4(3^n). \end{aligned}$$

It is now clear that 4 divides this quantity, since it divides both terms in the difference.

4. (a) Let $a, b \in \mathbb{R}$, with $a < b$. Then $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$, and since the rationals are dense in the real numbers, there exists an $r \in \mathbb{Q}$ so that $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$. Multiplying through by the positive number $\sqrt{2}$, we have $a < r\sqrt{2} < b$. Since $\sqrt{2}$ is irrational and r is rational, $r\sqrt{2}$ is irrational by a homework question. Thus, the irrational numbers are dense in the rational numbers.

   (b) Clearly a is an upper bound for that set. Let $\gamma < a$. Then since the rational numbers are dense in the real numbers, there exists an $r \in \mathbb{Q}$ so that $\gamma < r < a$. Thus $r \in \{s \in \mathbb{Q} | s < a\}$, and so γ is not an upper bound. By definition, $a = \sup\{s \in \mathbb{Q} | s < a\}$.

   (c) Since S and T are bounded, $\sup(S)$ and $\sup(T)$ are real numbers. Notice that for every $t \in T$, that $t \leq \sup(T)$, since $\sup(T)$ is an upper bound for T. Since $S \subseteq T$, for all $s \in S$ we have $s \leq \sup(T)$, so that $\sup(T)$ is an upper bound for S. Since $\sup(S)$ is the least of all upper bounds of S, and $\sup(T)$ is an upper bound of S, we have $\sup(S) \leq \sup(T)$.

5. (a) Since the rational numbers are dense in the real numbers, there exists rational numbers r_1 and r_2 so that $a < r_1 < b$, and $a < r_1 < r_2 < b$. Then the numbers $r_n = r_1 + \frac{1}{n}(r_2 - r_1)$ for $n \geq 3$ satisfy the following properties. First, $r_1 < r_n < r_2$ for all $n \geq 3$. Second, $r_n \in \mathbb{Q}$ for all $n \geq 3$. Lastly, $a < r_1 < r_{n+1} < r_n < r_2 < b$, so that we conclude that the sequence r_n is a collection of infinitely many distinct rational numbers strictly between a and b.

   (b) We create the mapping $1 \mapsto 0$, $2 \mapsto k$, $3 \mapsto -k$, $4 \mapsto 2k$, $5 \mapsto -2k$, etc., so that there is a mapping $f : \mathbb{N} \to k\mathbb{Z}$ given as

$$f(n) = \begin{cases} \frac{kn}{2} & \text{if } n \text{ is even} \\ \frac{-k(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Clearly the range of f is as listed, and that f is onto that set. Moreover, f is 1-1 since the only time outputs could equal each other is when they come from inputs of the same parity. And upon restriction to those inputs, the function is increasing, hence 1-1. So f is a bijection showing $\mathbb{N} \sim k\mathbb{Z}$.

(c) Suppose the irrational numbers are countable. Notice that $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$. Since the rational numbers are countable, then the real numbers are a countable (finite) union of countable sets, hence is countable. This is a contradiction to $\mathbb{R}$ being uncountable.