Math 331 Midterm #2B Solutions

A Leprechaun Stealing someone else's pot of gold!

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Hi kiddies! My name is Pat the Leprechaun, and I'm tired of just guarding my own pot of gold. So, this Saint Patrick's Day I thought I'd go cause a little mischief and steal someone else's pot of gold! Ha! Here are the exam solutions. ROCK ON!

1. (a) No, this isn't a basis. Any basis for $\mathbb{R}^3$ has 3 vectors in it... this set only has 2 vectors in it, so it can't be a basis.

   (b) Let $b = (b_1, b_2, b_3) \in \mathbb{R}^3$. We show that $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$ by solving the system of linear equations
   $$a_1v_1 + a_2v_2 + a_3v_3 = b\,.$$
   We write this in augmented matrix form as
   $$\left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 3 & -3 & 0 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 3 & -3 & 0 & b_3 \\ 0 & 0 & 1 & b_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & -9 & 0 & b_3 - 3b_1 \\ 0 & 0 & 1 & b_2 \end{array}\right]$$
   It is now clear that this system is consistent, so this set spans $\mathbb{R}^3$.

   (c) Since the dimension of $\mathbb{R}^3$ is 3, and this is a set of 3 vectors that spans $\mathbb{R}^3$, we know this is a basis.

2. A good number of people had the right idea for this problem, but I saw a lot of nonstandard and confusing notation often making the meaning of what people were saying unrecoverable. So there were a lot of people that missed points for a variety of reasons.

   The main notational problem that I'm talking about is as follows. Let's use part (b) as an example: it asks you to find a basis for P_2. That means that you should come up with a set of elements that are *in* P_2. Since P_2 is a set of polynomials, that means that you should display a set of polynomials. Some people gave me column vectors, containing numbers, column vectors containing x's, row vectors containing x, y and z's. None of these objects I just listed are polynomials. I also saw some people multiplying these "vectors" together, which is something that we would never do: remember, the only "multiplication" we do is by real numbers scaling vectors. The vectors themselves (polynomials, in this case) only add and subtract, and never multiply (at least not until you get into some higher linear and abstract algebra, with exception to the vector space $\mathbb{R}^3$ and the cross product $\times$.) I *strongly suggest* that anyone who missed a fair number of points should very carefully review their solutions in order to correct them. And I think your path to correction should be as follows: (1) Read what you wrote, and read Corey's comments. (2) Determine why what you wrote is incorrect, or why what you wrote doesn't make any sense. This could include expressing a polynomial as a column vector, or any of those that I mentioned above. (3) Write the problem on a piece of paper and wait at least one day. (4) Give yourself 7 minutes to do parts (a), (b) and (c) from this problem, closed book.

   For part (a) I also saw some people give examples of the phenomena they wanted to prove, not a proof (examples are good to get the idea of what is going on, but a proof would likely ask you to show that something always happens, no matter what example you choose).

   The point of this problem was to remind everyone that we have examples of vector (sub)spaces that are not always $\mathbb{R}^n$. The general feeling Corey got from grading was that a lot of people really want to treat every vector space as $\mathbb{R}^n$; this is not surprising, it is our most familiar example. We will see shortly that there is a connection between, for example, $\mathbb{R}^3$ and P_2. But this connection is a precise one which needs to be handled very carefully.

   Okay, here is the solution to this problem:

   (a) We use the test for a subspace. Let $p_1 = ax^2 + bx + c$ and let $p_2 = dx^2 + ex + f$ be elements of P_2. Then $p_1 + p_2 = (a + d)x^2 + (b + e)x + (c + f) \in P_2$. If α is a real scalar, then $cp_1 = \alpha ax^2 + \alpha bx + \alpha c \in P_2$. By the test for a subspace, we're done.

   (b) Let $\mathcal{B} = \{1, x, x^2\}$. Corey claims this is a basis for P_2.

   Linear independence: Suppose $a_1(1) + a_2(x) + a_3(x^2) = 0$, the zero polynomial. Then it is clear that $a_1 = a_2 = a_3 = 0$, since the only polynomial of degree less than or equal to 2 having more than 2 roots is the zero polynomial.

Spanning: Let $p = ax^2 + bx + c$ be an arbitrary polynomial in P_2. Then we have

$$a(x^2) + b(x) + c(1) = p\,.$$

(c) Our basis above has 3 elements in it, so the $\dim(P_2) = 3$.

3. (a) By adding row 2 to row 1 and row 3 to row 1, we get:

$$\begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & 9 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

Now, the reduced row-echelon form of a matrix requires that zeros are both above and below a 1 that starts any nonzero row. Just like the quiz, Corey won't be hard on you who row reduced as he did above.

(b) The row vectors $[1064], [0196]$, and $[0015]$ form a basis for the row space of A.

(c) Columns 1, 2, and 3 of A (not the reduced row echelon form of A) form a basis for the column space of A.

(d) The rank of A is 3.

(e) Since the number of columns $=$ Rank(A) + Nullity(A), we have $4 = 3 +$ Nullity(A), so the dimension of the null space is 1.

4. We show that the set of vectors is linearly independent. Since the dimension of $\mathbb{R}^2$ is 2, this will be enough to deduce that this set of 2 vectors is a basis. Suppose $a_1(8, 2) + a_2(1, -2) = 0$. Then this gives equations

$$\begin{aligned} 8a_1 + a_2 &= 0 \\ 2a_1 - 2a_2 &= 0 \end{aligned}$$

So $a_1 = a_2$ from the second equation, and by the first we see that $8a_1 + a_1 = 9a_1 = 0$, so $a_1 = 0$. This means, since $a_1 = a_2$, that $a_2 = 0$ as well.

Corey noticed some people who had difficulty showing this was a basis, mainly because they jumbled up Theorem 4.12 on page 223. Keep in mind that if you're going to apply a theorem you absolutely and without question need to check the hypotheses. So if you're going to use the theorem to show that the above is a basis, you'll at least have to mention that (1) The $\dim(\mathbb{R}^2) = 2$, (2) The set $\mathcal{B}$ is a set of 2 vectors, and then show (3) the set is linearly independent (or that it spans). THEN, you'll have to mention that we have a result which says that $\mathcal{B}$ is a basis.

(b) The transition matrix is $\begin{bmatrix} 8 & 1 \\ 2 & -2 \end{bmatrix}$.

It appeared, as Corey graded the exams, that people tried an alternative method that is outlined in the book, where you augment a matrix with the identity matrix, and then row reduce to get the desired matrix. That method is certainly sound, but most people who attempted this got the result backwards, and ended up with P^{-1}, rather than P. This would be the matrix that eats something written with respect to the standard basis and spits out the same vector with respect to the weird basis. For those still interested in trying this method, I strongly suggest that you take a close look at the theorem in the book, and keep track of which basis is which. Also, Corey will remind you of another method in class that makes it less likely that you will make this mistake.

(c) We send the vector $(3, 6)$ through the matrix P to get

$$\begin{bmatrix} 8 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ -6 \end{bmatrix}.$$

So $(3, 6)_{\mathcal{B}} = (30, -6)_{\mathcal{E}}$.

5. Each of the solutions to these are given below. Rock on!

(a) Let $u, v \in W$. Then $Au = Av = 0$. Then $A(u + v) = Au + Av = 0 + 0 = 0$. So $u + v \in W$. Let c be a real scalar. Then $A(cu) = cAu = c \cdot 0 = 0$. So $cu \in W$. Thus, by the test for a subspace, W is a subspace of V.

(b) We know that the dimension of $\mathbb{R}^n$ is n. So all we must show is that the set $\{Av_1, \ldots, Av_n\}$ is linearly independent. It will then follow by our favorite theorem that this is a basis for $\mathbb{R}^n$. We assume that A is invertible and that $\{v_1, \ldots, v_n\}$ is a basis. Hence, it's linearly independent.

So suppose that $\sum a_i(Av_i) = 0$. We want to show that $a_i = 0$ for all i. But

$$\sum a_i(Av_i) = \sum A(a_i v_i) = A\left(\sum a_i v_i\right) = 0\,.$$

Since A is invertible, we apply A^{-1} to both sides to get

$$\sum a_i v_i = A^{-1}0 = 0\,.$$

Since $\{v_1, \ldots, v_n\}$ is linearly independent, it follows that $a_i = 0$ for all i.

(c) We need to check that this set is linearly independent and that it spans the space in question: $Span(S)$. We have assumed that it is linearly independent in the statement of the problem. By definition, the set S is a spanning set for $Span(S)$. So the set $\{v_1, v_2, v_3\}$ is a basis for $Span(S)$. Thus the dimension of $Span(S)$ is 3.