

Midterm # 2 Solutions

Arrowhead Water Bottles

November 27, 2007

Hi everyone! I'm sitting on Corey's desk and since he seems to have so many visitors today, I'll be writing the exam solutions. Enjoy!

1. Here is the solution to the first problem. If you got the wrong antiderivative, then pay special attention to these steps!

$$\begin{aligned}\int_{y=0}^1 \int_{x=0}^1 \int_{z=0}^1 ye^{xy} dz dx dy &= \int_{y=0}^1 \int_{x=0}^1 zye^{xy} \Big|_{z=0}^1 dx dy \\ &= \int_{y=0}^1 \int_{x=0}^1 ye^{xy} dx dy \\ &= \int_{y=0}^1 e^{xy} \Big|_{x=0}^1 dy \\ &= \int_{y=0}^1 e^y - 1 dy \\ &= e^y - y \Big|_{y=0}^1 \\ &= (e^1 - 1) - (e^0 - 0) \\ &= e - 2.\end{aligned}$$

2. Many of you factored out the following expression, and so I do the same as I integrate:

$$\begin{aligned}\int_{x=0}^2 \int_{z=0}^x \int_{y=0}^z 2x(1-z^2) dy dz dx &= \int_{x=0}^2 \int_{z=0}^x 2xz(1-z^2) dz dx \\ &= \int_{x=0}^2 \int_{z=0}^x 2xz - 2xz^3 dz dx \\ &= \int_{x=0}^2 xz^2 - \frac{1}{2}xz^4 \Big|_{z=0}^x dx \\ &= \int_{x=0}^2 x^3 - \frac{1}{2}x^5 dx \\ &= \frac{1}{4}x^4 - \frac{1}{12}x^6 \Big|_{z=0}^2 \\ &= 4 - \frac{64}{12} \\ &= \frac{-16}{12} \\ &= -\frac{4}{3}.\end{aligned}$$

3. Here is the question that I had intended to ask. Let D be the solid in \mathbb{R}^3 bounded by the following equations: $x + y \leq 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + 2y + 3z = 6$. Compute the volume of D . We discussed in class why this differs slightly from the question that I actually asked you. The solution is as follows.

We describe the region as an elementary region by pointing out that $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$, and $0 \leq z \leq \frac{1}{3}(6 - x - 2y)$. So the volume is given by the integral:

$$\begin{aligned}
 \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{\frac{1}{3}(6-x-2y)} dz dy dx &= \int_{x=0}^1 \int_{y=0}^{1-x} \frac{1}{3}(6 - x - 2y) dy dx \\
 &= \int_{x=0}^1 \left(2 - \frac{x}{3}\right)y - \frac{y^2}{3} \Big|_{y=0}^{1-x} dx \\
 &= \int_{x=0}^1 \left(2 - \frac{x}{3}\right)(1 - x) - \frac{(1-x)^2}{3} dx \\
 &= \int_{x=0}^1 \frac{5}{3}(1 - x) dx \\
 &= \frac{5}{3}x - \frac{5}{6}x^2 \Big|_{x=0}^1 \\
 &= \frac{5}{6}.
 \end{aligned}$$

4. Since T is a function which is represented by matrix multiplication, and the determinant of the matrix is nonzero, we know that T sends parallelograms to parallelograms. So since $T(0, 0) = (0, 0)$, $T(1, 0) = (2, 1)$, $T(0, 1) = (1, -1)$ and $T(1, 1) = (3, 0)$, the image of T is the parallelogram traced out by these vertices.
5. This is the same example as given in class, except that θ ranges from 0 to π , rather than from 0 to $\pi/2$. So instead of tracing out a quarter circle, the image of T is a semi-circle. One can see this by fixing an angle θ , and noticing that as r goes from 0 to 1 the line that is formed traces out a line centered at the origin, heading off at a the angle θ , as measured from the positive x axis. The line ends once you are a distance 1 from the origin. Upon considering all angles θ from 0 to π , one can see that the upper-half disk is traced out.
6. The function T isn't one-to-one since $T(0, 0) = T(0, \pi) = (0, 0)$.