Exam # 1 Solutions

Sonic the Hedgehog

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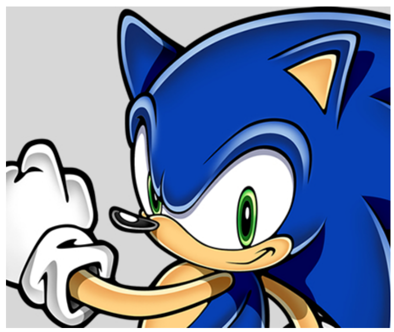

Hi kids! Sonic the Hedgehog here sharing with you the solutions from the last exam. Corey assures me that it has never taken him this long to grade a stack of exams. See, he's very thorough, and there are lots of details to check. And Corey checked them all, as he always does. Now, keep in mind that I'm just a Hedgehog that always seems to have huge gloves on, and so it's pretty hard to type a bunch of stuff out. So in many cases I've supressed the extra notation of expressing $\int_0^1$ instead of $\int_{x=0}^1$, for instance, when the order of integration is clear from context. In class, Corey has rightfully emphasized this subtlety, but it needn't be overdone here. I've taken a look at the exams myself and have some comments besides just the solutions, if you'd like to read these comments, you'll find them at the end. Oh, and ROCK ON!

1. Double integrals are awesome!

$$\begin{aligned}
\int_{x=0}^2 \int_{y=-1}^1 x^2y + y^2x\,dy\,dx &= \int_{x=0}^2 [x^2\tfrac{1}{2}y^2 + \tfrac{1}{3}y^3x|_{y=-1}^1]dx \\
&= \int_{x=0}^2 [x^2\tfrac{1}{2}(1)^2 + \tfrac{1}{3}(1)^3x] - [x^2\tfrac{1}{2}(-1)^2 + \tfrac{1}{3}(-1)^3x]dx \\
&= \int_{x=0}^2 \tfrac{2}{3}x\,dx \\
&= \tfrac{1}{3}x^2|_0^2 \\
&= \tfrac{4}{3}.
\end{aligned}$$

2. (a) Omitting the extra notation I typically use in class, I'll tell you here I'm integrating

with respect to y first, and we get

$$\begin{aligned}
\int\int f(x,y)dA &= \int_0^1 \int_{-1}^1 ye^x dydx \\
&= \int_0^1 \tfrac{1}{2}y^2 e^x|_{-1}^1 dx \\
&= \int_0^1 0dx \\
&= 0.
\end{aligned}$$

(b) We integrate with respect to x first to get

$$\begin{aligned}
\int\int_R f(x,y)dA &= \int_{-1}^1 \int_0^1 ye^x dxdy \\
&= \int_{-1}^1 ye^x|_0^1 dy \\
&= \int_{-1}^1 y(e-1)dy \\
&= \tfrac{1}{2}y^2(e-1)|_{-1}^1 \\
&= \tfrac{1}{2}[(e-1)-(e-1)] \\
&= 0.
\end{aligned}$$

3. Again keeping the cumbersome notation to a minimum, we integrate with respect to y first, although this is clear from context.

$$\begin{aligned}
\int_0^1 \int_0^{x^2} x^2 + xy - y^2 dydx &= \int_0^1 [x^2y + \tfrac{1}{2}xy^2 - \tfrac{1}{3}y^3|_0^{x^2}]dx \\
&= \int_0^1 x^4 + \tfrac{1}{2}x^5 - \tfrac{1}{3}x^6 dx \\
&= \tfrac{1}{5}x^5 + \tfrac{1}{12}x^6 - \tfrac{1}{21}x^7|_0^1 \\
&= \tfrac{1}{5} + \tfrac{1}{12} - \tfrac{1}{21}.
\end{aligned}$$

4. (a) The region described is y simple by noticing it is described by the inequalities $0 \le x \le 1$ and $x^2 \le y \le x$. So we integrate as follows, the order again clear from context:

$$\begin{aligned}
\int\int_D f(x,y)dA &= \int_0^1 \int_{x^2}^x x^2y\ dydx \\
&= \int_0^1 \tfrac{1}{2}x^2y^2|_{x^2}^x dx \\
&= \tfrac{1}{2}\int_0^1 x^4 - x^6 dx \\
&= \tfrac{1}{2}[\tfrac{1}{5}x^5 - \tfrac{1}{7}x^7]_0^1 \\
&= \tfrac{1}{35}.
\end{aligned}$$

(b) The region described is x-simple by noticing it is described by the inequalities $0 \le y \le 1$ and $y \le x \le \sqrt{y}$. So we integrate as follows, the order again clear from context:

$$\begin{aligned}
\int\int_D f(x,y)dA &= \int_0^1 \int_y^{\sqrt{y}} x^2y\ dxdy \\
&= \int_0^1 \tfrac{1}{3}x^3y|_y^{\sqrt{y}} dx \\
&= \tfrac{1}{3}\int_0^1 y^{5/2} - y^4 dy \\
&= \tfrac{1}{3}[\tfrac{2}{7}y^{7/2} - \tfrac{1}{5}y^5]|_0^1 \\
&= \tfrac{1}{35}.
\end{aligned}$$

5. As a super-quick hedgehog, I can quickly type out both parts, to help those people who may have tried part (b).

(a) The region described is given as an x-simple region, but can be described as a y-simple region as $0 \leq y \leq 1$, and $0 \leq x \leq \sqrt{1-y^2}$. So we may interchange the order of integration as follows:

$$\begin{aligned}
\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \sqrt{1-y^2}dydx &= \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} \sqrt{1-y^2}dxdy \\
&= \int_{y=0}^{1} x\sqrt{1-y^2}|_0^{\sqrt{1-y^2}} dy \\
&= \int_{y=0}^{1} 1-y^2dy \\
&= y - \tfrac{1}{3}y^3|_0^1 \\
&= 1-\tfrac{1}{3} \\
&= \tfrac{2}{3}.
\end{aligned}$$

(b) The integral is given as an integral over a y-simple region, but it appears to want to be integrated with respect to y first... i.e., it seems that we need to consider this region as an x-simple region. The reason the integral seems to want this is by observation. If we integrate with respect to x first, then it seems as though trig substitution is the only way to go. And Corey has told me many times that none of you seem to like trig substitution. However, there is a straightforward u-substitution if we integrate with respect to y first. As an x-simple region, we have $0 \leq x \leq 1$, and $0 \leq y \leq \sqrt{1-y^2}$. So we integrate as:

$$\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} 3y\sqrt{x^2+y^2}dxdy = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} 3y\sqrt{x^2+y^2}dydx.$$

Now to integrate the first integral (with respect to y), we treat x as a constant and set $u = x^2+y^2$. Then $du = 2ydy$, and so $\frac{3}{2}du = 3ydy$, and

$$\begin{aligned}
\int_{y=0}^{\sqrt{1-x^2}} 3y\sqrt{x^2+y^2}dydx &= \int_{y=0}^{\sqrt{1-x^2}} \tfrac{3}{2}\sqrt{u}du \\
&= (x^2+y^2)^{3/2}|_{y=0}^{\sqrt{1-x^2}} \\
&= 1-x^3.
\end{aligned}$$

So we have

$$\begin{aligned}
\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} 3y\sqrt{x^2+y^2}dydx &= \int_{x=0}^{1} 1-x^3dx \\
&= x-\tfrac{1}{4}x^4|_0^1 \\
&= 1-\tfrac{1}{4} \\
&= \tfrac{3}{4}.
\end{aligned}$$

Other comments. I have several comments about the exams. First, I would say that about 80% of the mistakes that Corey saw were not math 252 mistakes, but very basic algebra and math 212 types of mistakes. This is both a good and bad thing. It's good because it seems that a lot of you are getting the material from this course. But it's bad because there are many of you that lack the skills required to succeed throughout the rest of the course. If you found yourself making these sorts of mistakes you should know that these need to be fixed before the next midterm, and that there is no more class time to spend reviewing these concepts (but you can see Corey in his office hours and he'll be happy to discuss these matters there).

Another very curious thing that Corey saw on the exams was a conspicuous lack of "=" signs. It seems that many people would rather use $\Rightarrow$, or $\rightarrow$ instead. Corey has no idea where this habit started. You all should know that $\rightarrow$ usually signifies a limit existing in some sense, and that $\Rightarrow$ literally is used to mean "implies", a logical symbol which connects logical sentences. Neither of these symbols was used appropriately, and it blurs the meaning of what you write–sometimes in an unrecoverable way. Nobody on the exam seemed to blur this meaning too much, but anyone who does have this habit should know that it technically isn't correct, and somewhere, sometime, you could lose points over it, or even flunk a class because of it.

See, mathematics is a science... sometimes called the queen of the sciences. No other science could exist without mathematics since science measures quantity, an inherently mathematical concept. As a result, there is a very strict set of logical rules which govern mathematics, and this logical set of rules (logic, itself, actually) is not open to interpretation. In a very real sense, if anything about some mathematical argument is wrong, then the entire line of reasoning is wrong. "Partial credit" was invented by nice teachers.

And so I strongly urge all of you to learn to express yourselves clearly and meaningfully. This means that you should use = signs when appropriate, and to not use other symbols inappropriately before I decide to start taking points off. I am not talking about showing more of your work (all of the work you showed was great–perfect level of detail on all but a few exams), but to express the existing work in an appropriate framework with appropriate symbols. Corey doesn't have any plans to start grading your mathematical grammar, but he guarantees that the people in his Math 553 class have had enough of his "stylistic" comments, and suggests that you'll all be required to take more accountability for your work as time goes on. Quite frankly, the university has it more or less set up so that you can't really graduate without doing exactly that.

Most importantly, though, Corey understands that mathematics is difficult. And this class can be complicated at times. So having a certain way of doing something helps sometimes. Corey's just pointing out that certain methodologies will eventually lead to deep misunderstandings later on, and he wants to make sure you're aware of these matters. He also encourages you to look at all of the material Corey has ever produced that is mathematical in nature, as these may be helpful examples. This includes looking at his other course webpages, his old course webpages, even his old University of Oregon website (just Google his name), or his research publications.

All of that being said, for the most part, the exams were really good. After all, the average was 90%! So as I say, these mistakes are mostly just math 212 mistakes, but I really urge all of you who may have made mistakes of this sort to not take them lightly. And, of course, Rock on!