Math 213 Midterm # 2 Solutions

Evie Mae Dunn

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Hi everyone! I'm Corey's new niece, and I was born yesterday, March 4th, 2008! I sort of had a busy day yesterday at the hospital, and entertaining family. But today I managed to get away for a little bit and write your solutions today. Finally! Some alone time! And, of course, mathematics would be my choice of activity. ROCK ON!

1. (a) One can compute this limit using L'Hospital's rule:

$$\lim_{n\to\infty} \frac{4n^3}{2n^3+4n+1} = \lim_{x\to\infty} \frac{4x^3}{2x^3+4x+1} = \lim_{x\to\infty} \frac{12x^2}{6x^2+4} = \lim_{x\to\infty} \frac{24x}{12x} = 2$$

(b) This is the same question as was on the quiz. See the quiz solutions.

(c) For this one we notice $\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$. And so

$$\lim_{n\to\infty} \frac{n!}{(n+2)!} = \lim_{n\to\infty} \frac{1}{(n+1)(n+2)} = 0.$$

2. (a) We use the test for divergence to conclude this series diverges, since $\lim_{n\to\infty} \frac{n-1}{n+1} = 1 \neq 0$.

(b) We use the limit comparison test, with $b_n = \frac{1}{n}$. This is a valid comparison, since

$$\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

We know that $\sum \frac{1}{n}$ is a divergent series (it's a p-series, with $p = 1 \leq 1$), so $\sum \frac{1}{n+1}$ also diverges.

(c) This is the same convergent geometric series that was on the quiz.

(d) We may use a direct comparison to the convergent p-series $\sum \frac{1}{n^2}$, as $\frac{n^2}{n^4+5n+7} \leq \frac{n^2}{n^4} = \frac{1}{n^2}$. So this series converges.

(e) This is the same convergent series that was on the quiz. Keep in mind that this series is NOT geometric, but can be compared to a convergent geometric series.

3. The absolute value of each of the members gives us the series $\sum \frac{1}{\sqrt{n+1}}$, which may be compared via the limit comparison test to the divergent p-series $\sum \frac{1}{n^{1/2}}$. This comparison is valid since

$$\lim_{n\to\infty} \frac{1/\sqrt{n+1}}{1/n^{1/2}} = \lim_{n\to\infty} \frac{n^{1/2}}{(n+1)^{1/2}} = 1.$$

So this series is not *absolutely* convergent. But we notice that $a_{n+1} = \frac{1}{\sqrt{(n+1)+1}} \leq \frac{1}{\sqrt{n+1}} = a_n$, and that $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$. So the alternating series test applies and we conclude that $\sum \frac{(-1)^n}{\sqrt{n+1}}$ conditionally converges.

(b) We use the test for divergence and note that $\lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$. Thus the limit $\lim_{n\to\infty} \frac{(-1)^n n^3}{n^3+1}$ does not exist. Therefore by the test for divergence, this series diverges.

Remark. Remember that Corey made a big stink in class about how if $\lim_{n\to\infty} a_n = a > 0$, then $\lim_{n\to\infty}(-1)^n a_n$ was a limit which did not exist, namely because the even terms would tend towards a, and the odd terms would tend towards $-a$, and if $a \neq 0$, then this sequence would seem to approach two different values. This is not a proof, but justification why the limit above does not exist. A lot of you used the fact that $\lim_{n\to\infty} \frac{n^3}{n^3+1} = 1 \neq 0$ to conclude that the series diverges. But $\frac{n^3}{n^3+1}$ isn't what's being summed, there is a $(-1)^n$ in there, so strictly speaking it should be the latter limit which should be studied. Nobody missed points for not entertaining the $(-1)^n$, but I thought I would let everyone know that there is a little bit more work to be done when studying most of the solutions that Corey saw.

(c) We use the ratio test, for $a_n = \frac{n}{2^n}$:

$$\lim_{n\to\infty} \left| \frac{(n+1)/2^{n+1}}{n/2^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{2^n}{2^{n+1}} \right| = \frac{1}{2} < 1.$$

So we conclude that this series absolutely converges.

(d) Notice that $\sum \frac{1}{n^{1/3}}$ diverges, since it is a p-series with $p = \frac{1}{3} \leq 1$, so this series does not *absolutely* converge. We also notice that $a_{n+1} = \frac{1}{(n+1)^{1/3}} \leq \frac{1}{n^{1/3}} = a_n$, and that $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$. So the alternating series test applies and this series converges conditionally.

(e) We use the ratio test with $a_n = \frac{(-3)^n}{n!}$ to conclude that this series converges absolutely:

$$\lim_{n\to\infty} \left| \frac{(-3)^{n+1}/(n+1)!}{(-3)^n/n!} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1}}{(-3)^n} \cdot \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| (-3) \cdot \frac{1}{n+1} \right| = 0 < 1.$$