212 Midterm #2 Solutions

By: A Holiday Starbucks Card

November 14, 2006

Hello, potential consumers! I'm one of the many holiday gift cards that people are so obsessed with seeing earlier and earlier every year. Even though I myself get sick of seeing these decorations so much before Thanksgiving, I can't talk. I was in a Starbucks recently watching Corey grade your exams and decided it was time to let you all know the solutions. I can tell Corey is much too busy to type these things out all the time so I've taken the initiative and went ahead and posted this report. I hope it helps! The grade breakdown is below. ROCK ON!

Score	Number of People who Scored that
90-∞	5
80-89	4
70-79	5
60-69	1
$-\infty$ - 59	1

1. Okay, so this is another use of the 2nd Fundamental Theorem of Calculus. Even a gift card such as myself could get excited about this problem.

Pink Exam: Please find the derivative of $F(x)=\int_1^{e^{3x}}\sqrt{t^6+1}dt$.

$$\frac{d}{dx}\left[\int_1^{e^{3x}}\sqrt{t^6+1}dt\right]=\sqrt{(e^{3x})^6+1}\cdot\frac{d}{dx}\left[e^{3x}\right]=\sqrt{(e^{3x})^6+1}\cdot e^{3x}\cdot 3\,.$$

Blue Exam: Please find the derivative of $F(x)=\int_1^{e^{2x+1}}\sqrt{t^4+1}dt$.

$$\frac{d}{dx}\left[\int_1^{e^{2x+1}}\sqrt{t^4+1}dt\right]=\sqrt{(e^{2x+1})^4+1}\cdot\frac{d}{dx}\left[e^{2x+1}\right]=\sqrt{(e^{2x+1})^4+1}\cdot e^{2x+1}\cdot 2\,.$$

2. Please compute the derivative of the function $f(x)=\arctan(\sqrt{x})$.

$$\frac{d}{dx}[\arctan(\sqrt{x})]=\frac{1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}[\sqrt{x}]=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\,.$$

3. Pink Exam. Please compute $\int\frac{3x^2}{1+x^6}dx$. This is one we did in class as an arctangent integral. So we let $u=x^3$, and so $du=3x^2dx$. Then

$$\int\frac{3x^2}{1+x^6}dx=\int\frac{du}{1+u^2}=\arctan u+C=\arctan(x^3)+C\,.$$

Blue Exam. Please compute $\int\frac{1}{\sqrt{4-x^2}}dx$. Well, the first thing we do is recognize this as an arcsin integral, and attempt to factor a 4 out of the denominator:

$$\sqrt{4-x^2}=\sqrt{4\left(1-\frac{x^2}{4}\right)}=\sqrt{4}\sqrt{1-\frac{x^2}{4}}=2\sqrt{1-\left(\frac{x}{2}\right)^2}\,.$$

We conclude that $\int\frac{1}{\sqrt{4-x^2}}dx=\frac{1}{2}\int\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}dx$. Let $u=x/2$, and so $du=\frac{1}{2}dx$, or $2du=dx$. Then we have

$$\frac{1}{2}\int\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}dx=\frac{2}{2}\int\frac{1}{\sqrt{1-u^2}}du=\arcsin u+C=\arcsin\left(\frac{x}{2}\right)+C\,.$$

4. Pink Exam. There were a few ways to do this one, I think the easiest is by u-substitution. Of course, if your solution differs from mine keep in mind that by some algebra your solution and mine may actually be the same. Also keep in mind that I am a small plastic card and may have different integration methods than you–nothing you wouldn't have seen in class, though. Let $u=x-1$. Then $du=dx$, and in addition $u+1=x$. We will need these to substitute all of the x's out:

$$\int x\sqrt{x-1}dx=\int x\sqrt{u}du=\int(u+1)\sqrt{u}du=\int(u^{3/2}+u^{1/2})du\,.$$

Integrating these separately, we get

$$\int (u^{3/2} + u^{1/2})du = \frac{1}{5/2}u^{5/2} + \frac{1}{3/2}u^{3/2} + C = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C\,.$$

<u>Blue Exam.</u> For this, we use u-substitution, with $u = e^x$. Then $du = e^x dx$. Our integral becomes:

$$\int e^x\sqrt{e^x - 1}dx = \int \sqrt{u}du = \frac{1}{3/2}u^{3/2} + C = \frac{2}{3}(e^x - 1)^{3/2} + C\,.$$

5. <u>Pink Exam.</u> Since this is a definite integral, there are two parts to this one. First we have to find an antiderivative, then plug in some numbers according to the 1st fundamental theorem of calculus. For the antiderivative we use integration by parts. We start with $u = x$ and $dv = \cos x dx$. So $du = dx$, and $v = \sin x$. Then according to integration by parts,

$$\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x\,.$$

So

$$\begin{aligned} \int_0^\pi x\cos x dx &= [x\sin x + \cos x]_0^\pi \\ &= (\pi\sin\pi + \cos\pi) - (0\sin 0 + \cos 0) \\ &= (-1) - (1) = -1 - 1 = -2\,. \end{aligned}$$

<u>Blue Exam.</u> Since this is a definite integral, we'll find an antiderivative first, then plug'n'chug á la the 1st fundamental theorem of calculus. We'll use integration by parts for the antiderivative. Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. So we get

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x\,.$$

So

$$\int_0^7 xe^x dx = [xe^x - e^x]_0^7 = (7e^7 - e^7) - (0e^0 - e^0) = 7e^7 - e^7 + 1\,.$$

6. For this problem we use substitution and then integration by parts. Won't this be fun? Let $w = \sqrt{x}$. Then $dw = \frac{1}{2\sqrt{x}}dx$, or $2\sqrt{x}dw = dx$. Of course, we'll have to substitute out that x in the expression for dx and so we write

$$\int \sin\sqrt{x}dx = \int 2w\sin w dw\,.$$

Now we use integration by parts. Let $u = 2w$ and let $dv = \sin x dx$. A few of you missed points because you factored the 2 out and then forgot about it. This is why I keep it in here because I'm just a silly plastic card worth only as much as I'm told–I can't always remember everything I do. Keeping the constants around is a little trick I picked up in my years of doing integrals, but it's just a minor suggestion for people who have trouble with those sorts of things. Anyway, we get $du = 2dw$, and $v = -\cos x$, and

$$\int 2w \sin w dw = 2w(-\cos w) - \int 2(-\cos w)dw = -2w \cos w + 2\sin w + C\,.$$

So we put it all together and get

$$\int \sin(\sqrt{x})dx = -2(\sqrt{x})\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C\,.$$

7. Pink Exam. We will use integration by parts twice. Let $u = e^x$, and $dv = \cos x dx$. Then $du = e^x dx$ and $v = \sin x$. So

$$\int e^x \cos x dx = e^x \sin x - \int \sin x e^x dx\,.$$

Now we work on the integral on the above right. Let $u = e^x$, and $dv = \sin x dx$. Then $du = e^x dx$ and $v = -\cos x$. So

$$\int \sin x e^x dx = e^x(-\cos x) - \int (-\cos x)e^x dx = -e^x \cos x + \int e^x \cos x dx\,.$$

Now at this point, if I wasn't being careful with my signs I might mess up and make a mistake and think something is zero when it's not. So pay attention to the signs as I work through the rest of this part of the problem:

$$\begin{aligned} \int e^x \cos x dx &= e^x \sin x - \int \sin x e^x dx \\ &= e^x \sin x - (-e^x \cos x + \int e^x \cos x dx) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x dx \end{aligned}$$

Okay, now we add $\int e^x \cos x dx$ to both sides and get

$$\begin{aligned} 2\int e^x \cos x dx &= e^x \sin x + e^x \cos x, \text{ so} \\ \int e^x \cos x dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + C \end{aligned}$$

Blue Exam. We will use integration by parts twice. Let $u = e^x$, and $dv = \sin x dx$. Then $du = e^x dx$ and $v = -\cos x$. So

$$\int e^x \sin x dx = -e^x \cos x + \int \cos x e^x dx\,.$$

Now we work on the integral on the above right. Let $u = e^x$, and $dv = \cos x dx$. Then $du = e^x dx$ and $v = \sin x$. So

$$\int \cos x e^x dx = e^x(\sin x) - \int (\sin x) e^x dx = e^x \sin x - \int e^x \sin x dx\,.$$

Now at this point, if I wasn't being careful with my signs I might mess up and make a mistake and think something is zero when it's not. So pay attention to the signs as I work through the rest of this part of the problem:

$$\begin{aligned} \int e^x \sin x dx &= -e^x \cos x + \int \cos x e^x dx \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x dx\,. \end{aligned}$$

Okay, now we add $\int e^x \sin x dx$ to both sides and get

$$\begin{aligned} 2\int e^x \sin x dx &= e^x \sin x - e^x \cos x, \text{ so} \\ \int e^x \sin x dx &= \frac{1}{2}(e^x \sin x - e^x \cos x) + C \end{aligned}$$

ROCK ON!!!